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Difference Divisor Graph of the Finite Group

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A B S T R A C T

Let $(Z_n, \oplus)$ be a finite group of integers modulo n and D_n a non-empty subset of Z_n containing proper devisors of n. In this paper, we have introduced difference divisor graph $Dif\,(Z_n, D_n)$ associated with Z_n whose vertices coincide with Z_n such that two distinct vertices are adjacent if and only if either $a-b \in D_n$, or, $b-a \in D_n$. Then we have investigated its algebraic and graph theoretic properties. Further, we have proved that the difference divisor graph $Dif\,(Z_n, D_n)$ is not a Cayley graph.

Keywords: Divisor graph, Difference divisor graph, Hamilton cycle, GCD-Graph, Cayley graph.

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1. Introduction

The notion of divisibility is an important tool in the study of theory of numbers and theory of algebra. The proper divisor of a positive integer n is denoted by d; the number of divisors of n is denoted by the arithmetic function $d(n)$ and the divisor function of n. Here $d \mid n$ but $d \neq n$ and let us agree upon $d \mid 0$ that is $d \neq 0$. For an integer $n \geq 1$, we denote by $Z_n = \{0, 1, ..., n-1\}$ the additive group, respectively and the ring of integers modulo n.

In recent years, divisors of a positive integer have played a central role in the study of arithmetic graphs. For instance, divisor graphs [1], Graph of Divisor function [2], zero divisor graphs [3, 4], and Cayley divisor graph [5] are few examples of arithmetic graphs associated with divisors of

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positive integers. There is another branch of graph theory called algebraic graph theory. It is one of the modern and active branches of pure mathematics, and it is mainly concerned with the properties of algebraic structures through graph structures. In recent years, algebraic graphical methods are likewise being utilized as a part of a few territories of engineering science, see [6, 7]. In particular, there has been a nearby creation between algebraic graph and finite group theories for over a century. Clean examinations and stunning outcomes from finite group theory have been demonstrated through combinatorial properties of algebraic graphs and the other way around [8, 9].

The algebraic structure Z_n is very eminent algebraic tool for constructing modern algebraic graphs. For such kind of study, some researchers defined algebraic graphs whose vertices are sets of elements of Z_n and edges are defied with respect to a condition on the divisors of n. This serves as the underlined motivation for this work the *GCD-graphs* to finite rings Z_n was first associated by Koltz and Sander for various values of $n>1$ [10]. Given $n>1$ and a non-empty subset $D_n \subset Z_n$, the gcd-graph denoted by $X_n(D_n)$ is defined as follows: each vertex is an element of Z_n with two distinct vertices a and b are adjacent if and only if $\gcd(a-b, n) \in D_n$, where they computed mainly Eigen values of $X_n(D_n)$. Madhavi introduced another class of algebraic graph of divisor function [11], called divisor Cayley graph $Cay(Z_n, S_n^*)$ that is the undirected simple graph with vertices Z_n, and for distinct vertices a and b are adjacent if and only if either $a-b \in S_n^*$, or, $b-a \in S_n^*$, where $S_n^* = D_n \cup (-D_n)$. Further, the investigation of $Cay(Z_n, S_n^*)$ was done by Chalapathi et al. [5]. Recently, Chalapathi and Kiran [12] introduced the order divisor graphs of subgroups of finite groups and studied its structural properties. For basic terminology and notations in algebraic graph theory we refer to [13], and for number theoretic properties of arithmetic functions we refer to [14].

2. Difference Divisor Graph

For any positive integer $n>1$, we denote D_n as a set of proper divisors of n. In this section, we define difference divisor graph and study its properties associated with finite group Z_n. Note that $D_n \subset Z_n$ and $|D_n| = d(n)-1$, where $d(n)$ is divisor function of n.

Definition 2.1 Let $n>1$ be a positive integer, then we define difference divisor graph

$Dif(Z_n, D_n)$ associated with a finite group Z_n as the undirected simple graph whose set of vertices coincides with Z_n such that two distinct vertices a and b are adjacent if and only if either $a-b \in D_n$, or, $b-a \in D_n$.

Throughout this paper n will always denote a positive integer and $n>1$. Now to justify our claim that the difference divisor graph is new, we as well illustrates by the following example how it is different from well-known algebraic graphs associated with group Z_n, refer [10 and 11].

Example 2.2 Let $n=9$, then $Z_9=\{0,1,2,...,8\}$ be the group of integers modulo 9, here

$D_9=\{1,3\}$ and $S_9^* = D_9 \cup (-D_9) = \{1, 3, 6, 8\}$. Clearly, the GCD -graph $X(D_n)$ and divisor Cayley-graph $Cay(Z_n, S_n^*)$ are both different form difference divisor graph $Dif(Z_n, D_n)$.

Theorem 2.3 For each $n>1$, the graph $Dif(Z_n, D_n)$ is connected.

Proof. Since $(a-1)-(a-2)=1\in D_n$ for every $a\in Z_n$, so the vertex $(a-1)$ is adjacent to the consecutive vertex $(a-2)$ in the graph $Dif(Z_n, D_n)$. Thus, there exist a path $0-1-2-\cdots-(n-2)-(n-1)$ between the vertices 0 and $n-1$ in $Dif(Z_n, D_n)$, and hence $Dif(Z_n, D_n)$ is connected.

Theorem 2.4 The graph $Dif(Z_n, D_n)$ is a tree if and only if n is prime.

Proof. Suppose $Dif(Z_n, D_n)$ is a tree of order n. Then it is acyclic graph of size $n-1$. Assume that n is not a prime, there exist a proper divisor d of n such that either $d-(-d)=0$ or $d-(-d)\neq 0$ in the group Z_n. So, we consider the following two cases.

Case 1. If $d-(-d)=0$, then $d=\dfrac{n}{2}\in Z_n$ if and only if n is even, otherwise d must be zero. Thus $\dfrac{n}{2}-\left(\dfrac{n}{2}-1\right), \left(\dfrac{n}{2}+1\right)-\dfrac{n}{2}$, and $\left(\dfrac{n}{2}+1\right)-\left(\dfrac{n}{2}-1\right)$ are the elements in D_n, so the pairs $\left(\dfrac{n}{2}-1, \dfrac{n}{2}\right)$, $\left(\dfrac{n}{2}, \dfrac{n}{2}+1\right)$, and $\left(\dfrac{n}{2}+1, \dfrac{n}{2}-1\right)$ are edges of the graph $Dif(Z_n, D_n)$. This means that $C_3:\left(\dfrac{n}{2}-1\right)—\dfrac{n}{2}—\left(\dfrac{n}{2}+1\right)—\left(\dfrac{n}{2}-1\right)$ is a cycle of length 3 in $Dif(Z_n, D_n)$, which is a contradiction to our hypothesis that $Dif(Z_n, D_n)$ is an acyclic graph.

Case 2. If $d-(-d)\neq 0$, then $d\neq\dfrac{n}{2}$. So, there exist at least three vertices $\dfrac{n}{2}+1$, $\dfrac{n}{2}+2$, and $\dfrac{n}{2}+3$ in $Dif(Z_n, D_n)$ such that the vertex sequence $\left(\dfrac{n}{2}+2\right)—\left(\dfrac{n}{2}+1\right)—\left(\dfrac{n}{2}+3\right)—\left(\dfrac{n}{2}+2\right)$ is a cycle in $Dif(Z_n, D_n)$, which is again a contradiction. From Case 1 and Case 2 we conclude that n must be a prime number.

Conversely, suppose that n is prime. For any $a\neq 0$ in Z_n, we have $o(ka)=o(a)=n$ for all $k\in Z_n-\{0\}$. It is clear that $o(a-b)=n$ for every $a, b\in Z_n$ and $a-b=1$ if and only if a and b

are consecutive elements in Z_n. Since n is prime; therefore, the vertex 0 is not adjacent to the vertex $n-1$ in $Dif\ (Z_n, D_n)$ because $n-1 \notin D_n$ if and only if n is prime. Hence, $Dif\ (Z_n, D_n)$ is a tree of order n.

Theorem 2.5 The graph $Dif\ (Z_n, D_n)$ is not complete if and only if $n > 2$.

Proof. Follows that a well-known observation that $n-1 \notin D_n$ if and only if $n > 2$, i.e. the vertex 0 is not adjacent to the vertex $n-1$ in $Dif\ (Z_n, D_n)$ if and only if $n > 2$.

Corollary 2.6 The graph $Dif\ (Z_n, D_n)$ is complete if and only if $n = 2$.

Proof. Suppose that $Dif\ (Z_n, D_n)$ is a complete graph. Then any two vertices in Z_n are adjacent in $Dif\ (Z_n, D_n)$. To show that $n = 2$, let if $n \neq 2$, then let n be 3 and clearly the vertices $0, 1, 2 \in V\left(Dif\ (Z_n, D_n)\right)$; by the Theorem 2.4, $Dif\,(Z_n, D_n)$ is a tree; it is not a complete graph. It follows that n cannot be 3. Further, if $n > 3$, then clearly, by the Theorem 2.5, $Dif\ (Z_n, D_n)$ is not complete. Thus, n cannot be greater than or equal to 3. Hence n must be 2.

Sufficiency. The sufficiency is clear to see as for instance, $Dif\ (Z_2, D_2) \cong K_2$, complete graph of order 2.

Theorem 2.7 If $n > 1$ is odd, then $Dif\ (Z_n, D_n)$ is a bipartite graph.

Proof. Since $n > 1$ is odd, each proper divisor of must be odd, thus, no two odd labeled vertices are adjacent and similarly no two even labeled vertices are adjacent in $Dif\,(Z_n, D_n)$. This implies that the set E_n of even labeled vertices and the set O_n of odd labeled vertices form a bipartition (E_n, O_n) of the vertex set Z_n in the graph $Dif\ (Z_n, D_n)$ and hence it is a bipartite graph.

Remark 2.8 The graph $Dif\ (Z_n, D_n)$ is bipartite but not complete for any odd $n > 1$.

Theorem 2.9 If n is even, then $Dif\ (Z_n, D_n)$ is a not a bipartite graph.

Proof. Assume that n is even. Suppose, if $Dif\ (Z_n, D_n)$ is bipartite graph. Then there exist a bipartition (A_n, B_n) where A_n and B_n are the set of even and odd integers of Z_n, respectively. Without loss of generality we may assume that $0 \in A_n$ and $n-1 \in B_n$. Since n is even, there exist a vertex $\frac{n}{2}$ in Z_n. Clearly, the vertices $\frac{n}{2}-1$ and $\frac{n}{2}+1$ are in B_n and the vertex $\frac{n}{2}$ in A_n. Therefore, the triad $\left(\frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+1\right)$ form a triangle in $Dif\ (Z_n, D_n)$. This violates the condition of bipartition. So our assumption is not true and hence $Dif\ (Z_n, D_n)$ is not a bipartite graph.

It should be appointed that the number of distinct edges in a path of simple graph G is called a length of the path. The distance $d(x, y)$ of vertices x and y of a graph G is the length of a shortest x, y-path. The diameter of G is the maximum distance of any two vertices of G may have and it is denoted by $diam(G)$. Similarly, the girth of G is denoted by $\mathrm{gir}(G)$ and defined as the length of the smallest cycle in G. Note that $\mathrm{gir}(G) = \infty$ if G is an acyclic graph.

Theorem 2.10 [19]. A tree with n vertices has diameter $n-1$.

Theorem 2.11 For $n \geq 2$ we have

$$diam\big(Dif\,(Z_n, D_n)\big) = \begin{cases} 1 & \text{if } n = 2 \\ n-1 & \text{if } n = p, \text{ an odd prime} \\ 3 & \text{if } n \geq 4 \text{ is even} \\ 4 & \text{if } n = p^{\alpha}, \ \alpha > 1 \text{ is positive integer} \end{cases}$$

Proof. If $n = 2$, then by the Corollary 2.6, $Dif\,(Z_2, D_2) \cong K_2$ is a complete graph, which has diameter 1. In other case, by the Theorem 2.5, $Dif\,(Z_n, D_n)$ is not complete, so that $diam\big(Dif\,(Z_n, D_n)\big) > 1$**.** If $n = p$ is an odd prime, then by the Theorem 2.4, $Dif\,(Z_n, D_n)$ is a tree with pendent vertices 0 and $n-1$. So, in view of Theorem 2.10, has diameter $n-1$.

Suppose $n \geq 4$ is a positive even integer. Then there exists a vertex $\frac{n}{2}$ in $Dif\,(Z_n, D_n)$ such that $\frac{n}{2}$ is adjacent to the other vertices $0, n-2$ and $n-1$. So, there exist a path $0 — \frac{n}{2} — (n-2) — (n-1)$ of length 3, which is smallest since the vertex 0 is not adjacent to the vertex $n-1$. So, in this case $diam\big(Dif\,(Z_n, D_n)\big)$ is 3. Finally, we consider the case when n is odd but not prime. But the vertices 0 and $n-1$ of $Dif\,(Z_n, D_n)$ are not adjacent and $\deg(0) = \deg(n-1) = |D_n|$. Also, 0 and $n-1$ have no common neighbor, it is clear that $diam\big(Dif\,(Z_n, D_n)\big) \geq d(0, n-1) \geq 4$. Now to show that $d(0, n-1) \leq 4$ for $n = p^{\alpha}$, $\alpha > 1$. We know that the set of proper divisors of p^{α} is $D_{p^{\alpha}} = \{1, p, p^2, ..., p^{\alpha-1}\}$ and $p^{\alpha-1} - 0$, $2p^{\alpha-1} - p^{\alpha-1}$, $2p^{\alpha-1} - (p^{\alpha} - 2)$, $(p^{\alpha} - 1) - (p^{\alpha} - 2)$ are in $D_{p^{\alpha}}$. Then by the definition of difference divisor graph, there exist a smallest path $0 — p^{\alpha-1} — 2p^{\alpha-1} — (p^{\alpha} - 2) — (p^{\alpha} - 1)$ of length 4 because $Dif\,(Z_{p^{\alpha}}, D_{p^{\alpha}})$ is a triangle free graph. This shows that $d(0, p^{\alpha} - 1) \leq 4$. Hence, in this case, $diam\big(Dif\,(Z_n, D_n)\big)$ is 4 for $n = p^{\alpha}$.

Theorem 2.12 The girth of the difference divisor graph is given by

$$\mathrm{gir}\left(Dif\ (Z_n, D_n)\right) = \begin{cases} \infty & \text{if n is a prime number} \\ 3 & \text{if } n \text{ is an even number} \\ 4 & \text{if } n \text{ is odd but not a prime number} \end{cases}.$$

Proof. Suppose n is prime, then by the Theorem 2.4, $Dif\ (Z_n, D_n)$ is acyclic graph, and thus girth of $Dif\ (Z_n, D_n)$ is infinity. Next, suppose n is even and $n > 2$, then in view of Theorem 2.9, $Dif\ (Z_n, D_n)$ is not a bipartite graph, so there exist an odd cycle in $Dif\ (Z_n, D_n)$. Thus, we always have a 3-cycle, namely $(n-1) — (n-2) — (n-3) — (n-1)$, which is smallest in $Dif\ (Z_n, D_n)$. Hence, $\mathrm{gir}\left(Dif\ (Z_n, D_n)\right) = 3$. Finally, suppose n is odd but not prime. Then by the Theorem 2.11, $Dif\ (Z_n, D_n)$ is a triangle free graph, and so we obtain a smallest cycle $(n-1) — (n-2) — (n-3) — (n-4) — (n-1)$ of length 4 in $Dif\ (Z_n, D_n)$. Thus, the girth of $Dif\ (Z_n, D_n)$ is equal to 4.

The investigations on group theory via combinatorial properties of graphs giving amazing results found in [15, 16]. This results as essential motivation for this work. The algebraic graphs namely, "Cayley graphs" associated to finite groups was originated by the Arthur Cayley [17]. Given a finite group X and a non-empty generating symmetric subset $S \subseteq X$, the Cayley graph denoted by $Cay\left(X, S\right)$ is defined as each vertex is an element of X with two vertices x, $y \in V\ (Cay\left(X, S\right))$ being adjacent if either xy^{-1} or $yx^{-1} \in S$, see [18].

In view of above investigations on Cayley graphs, we shall prove an important result that $Dif\ (Z_n, D_n)$ is not a Cayley graph.

Theorem 2.13 Let $n > 1$ be a positive integer. Then $Dif\ (Z_n, D_n)$ is not a Cayley graph.

Proof. Suppose $Dif\ (Z_n, D_n)$ is Cayley graph of the finite group Z_n. Then the subset D_n in Z_n is a symmetric set. By the definition of symmetric set, for every $d \in D_n \Rightarrow -d \in D_n$. This shows that the structure $(D_n, \oplus)$ forms an abelian subgroup of $(Z_n, \oplus)$, which is not true because sum of two proper divisors of $n > 1$ may not a proper divisor.

2. Traversable Properties of Difference Divisor Graph

A simple graph is said to be traversable if there exists a path between all the vertices without retracing the same path. In this section, we describe some categories like Eularian path and Hamiltonian path based on this path, and hence it shows that the difference divisor graph $Dif\ (Z_n, D_n)$ is Eularian and Hamiltonian for various values of $n > 1$.

Theorem 3.1 For any positive integer $n > 1$, the difference divisor graph $Dif\ (Z_n, D_n)$ is not Eularian.

Proof. By the definition of difference divisor graph, $Dif\ (Z_n, D_n)$ is not regular if and only if $n > 2$. So, clearly, $\deg(n-1) = |D_n|$ and $\deg(n-2) = |D_n| + 1$. Now there are two possibilities; if $|D_n|$ is even, then $|D_n| + 1$ is odd. On the other hand, if $|D_n|$ is odd, then $|D_n| + 1$ is even. Hence in both the possibilities, we found that degree of each vertex cannot be even, hence by the characterization of Eularian graphs [19], the graph $Dif\ (Z_n, D_n)$ is not an Eularian.

Theorem 3.2 Let $n > 2$ be even. Then the graph $Dif\ (Z_n, D_n)$ is Hamiltonian.

Proof. Let $n > 2$ be an even positive integer, we shall now show that $Dif\ (Z_n, D_n)$ is Hamiltonian. For thus we shall show that $Dif\ (Z_n, D_n)$ has a cycle that visits every vertex exactly once. To do this, let i be the i^{th} vertex in $Dif\ (Z_n, D_n)$ defined as $i \in \{0, 1, 2, ..., n-1\}$. Since n is even, $\frac{n}{2}$ exists and also $\frac{n}{2} \in D_n$; therefore, the pairs $(0, 1), (1, 2), \ldots, \left(\frac{n}{2}-1, n-1\right), (n-1, n-2), (n-2, n-3), \ldots \left(\frac{n}{2}, \frac{n}{2}-1\right)$, and $\left(\frac{n}{2}, 0\right)$ are adjacent in $Dif\,(Z_n, D_n)$. So, we construct a cycle $0-1-2 - \cdots - \left(\frac{n}{2}-1\right) - (n-1) - (n-2) - \cdots - \left(\frac{n}{2}+1\right) - \frac{n}{2} - 0$ which covers all the vertices in $Dif\ (Z_n, D_n)$. Thus there exist a cycle of length n that visits each vertex in $Dif\ (Z_n, D_n)$, and hence the graph is Hamiltonian.

Theorem 3.3 Let $n > 2$ be odd. Then $Dif\ (Z_n, D_n)$ is not Hamiltonian.

Proof. Let $d(n)$ be the number of divisors of $n > 2$. We shall now show that $Dif\ (Z_n, D_n)$ is not Hamiltonian. If possible assume that $Dif\ (Z_n, D_n)$ is Hamiltonian, then by characterization of Hamiltonian graphs [19] for any two vertices a and b in $Dif\ (Z_n, D_n)$, the following in equality holds:

$$\deg(a) + \deg(b) \geq n.$$

Without loss of generality, we may assume that $a = 0$ and $b = n-1$ are two vertices of the graph $Dif\ (Z_n, D_n)$ for any positive odd integer $n > 2$. By the definition of difference divisor graph, we have $\deg(0) = \deg(n-1) = |D_n|$.Therefore, $2|D_n| \geq n \Rightarrow 2(d(n)-1) \geq n \Rightarrow d(n) \geq \frac{n+2}{2}$, which is not true for any positive odd integer $n > 2$. Hence $Dif\ (Z_n, D_n)$ is not Hamiltonian.

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