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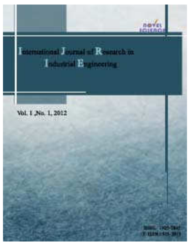

A New Method for Ranking Efficient DMUs Based on TOPSIS and Virtual DMUs

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ABSTRACT

In this paper, a method for ranking efficient $DMUs$ based on TOPSIS has been proposed. The difference between the distance of the center of gravity of all efficient $DMUs$ to the ideal point and the anti-ideal point after and before deleting efficient $DMUs$ one by one is the criteria of ranking efficient $DMUs$. In this paper, the proposed method is compared with AP (input oriented), MAJ (input oriented), AP (output oriented), MAJ (output oriented) models and norml_1 method. This comparison shows that the proposed method is better than the above-mentioned models. The proposed method is also always feasible and simpler in comparison with other methods.

1. Introduction

Data Envelopment Analysis (DEA) is a methodology for estimating the efficiency of decision making units ($DMUs$) with multiple inputs and outputs. For the first time, Charnes et al. [1], introduced a method for determining the efficiency of decision making units. In 1993, Anderson and Peterson [2] proposed the super-efficiency method for ranking efficient DMUs (specifying the best DMU among all efficient $DMUs$). Mehrabian et al. (MAJ) [3] have modified the AP model. Jahanshaloo et al. [4] presented the method for ranking efficient $DMUs$ by $l_1$norm. Also Zhu presented a ranking method, named PCA [5]. The TOPSIS (technique for order performance (preference) by similarity to ideal solution) is a method for ranking $DMUs$ which was presented by Hwang & Yoon (1981) [6]. TOPSIS ranks $DMUs$ by calculating the distances of $DMUs$ to the ideal point and the anti-ideal point, in which each DMU with maximum distance to the ideal point and minimum distance to anti-ideal point receives rank 1 and, every DMU with minimum distance to the ideal point and maximum distance to anti-ideal point receives rank n. This paper is organized into following sections: Insection 2, we provide an overview of data envelopment analysis and ranking models. In section 3, we introduce an approach for ranking efficient $DMUs$, then a numerical example is given in section 4, and finally in section 5 conclusions are drawn.

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2. An overview of Data Envelopment Analysis, ranking models and TOPSIS

2.1. CCR model

Data Envelopment Analysis is a method which is applied for obtaining efficiency of DMUs. Let us have **n** DMUs which have **m** inputs and **s** outputs, suppose$\{x_{1j}, \mathrm{x}_{2j}, \dots, \mathrm{x}_{mj}\}$ and $\{y_{1j}, \mathrm{y}_{2j}, \dots, \mathrm{y}_{sj}\}$ represent input and output vectors of DMU_j, respectively. For evaluation of the efficiency of DMU_o, CCR model is given below:

$$MAX \sum_{r=1}^{s} u_r\, y_{ro}$$

$$s.t. \textstyle\sum_{i=1}^{m} v_i\, x_{io} = 1$$

$$\textstyle\sum_{r=1}^{s} u_r\, y_{rj} - \sum_{i=1}^{m} v_i\, x_{ij} \le o \quad , j = 1, \dots, n \tag{1}$$

$$u_r \ge o \quad , r = 1, \dots, s \quad , \quad v_i \ge o \quad , i = 1, \dots, m$$

where, v_i and u_r represent the weights of i^th^ input and r^th^ output, respectively.

2.2.AP model

Anderson and Peterson [2] proposed Super Efficiency model. This model ranks DMUs by deleting DMU under evaluation from Production Possible Set (PPS) and offers the DEA model for other$DMUs$.

$$Max \sum_{r=1}^{s} u_r\, y_{ro}$$

$$s.t. \sum_{i=1}^{m} v_i\, x_{io} = 1$$

$$\textstyle\sum_{r=1}^{s} u_r\, y_{rj} - \sum_{i=1}^{m} v_i\, x_{ij} \le 0 \quad , j = 1, \dots, n \,, j \ne o \tag{2}$$

$$u_r \ge o \quad , r = 1, \dots, s \quad , \quad v_i \ge o \quad , i = 1, \dots, m$$

2.3.MAJ model

To eliminate the major problems of AP model, Mehrabian et al [3], proposed another model for ranking efficient DMUs. Their proposed model follows here:

$$Min\; 1 + w$$

$$s.t. \sum_{j=1, j\neq o}^{n} \mu_j x_{ij} \leq x_{io} + w \quad , i = 1, \dots, m$$

$$\sum_{j=1, j\neq o}^{n} \mu_j y_{rj} \geq y_{ro} \quad , r = 1, \dots, s \tag{3}$$

$$\mu_j \geq 0 \quad , \quad j = 1, \dots, n \quad , j \neq o$$

2.4.Rank by l_1 norm

Jahanshaloo et al [4], used l_1 norm for ranking extremely efficient $DMUs$. This model comes next:

$$Min \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + \propto$$

$$s.t. \sum_{j=1, j\neq o}^{n} \mu_j x_{ij} \leq x_i + w \quad , i = 1, \dots, m$$

$$\sum_{j=1, j\neq o}^{n} \mu_j y_{rj} \geq y_r \quad , r = 1, \dots, s \tag{4}$$

$$x_{\mathrm{i}} \geq \mathrm{x}_{\mathrm{io}} \quad , \quad \mathrm{i} = 1, \dots, \mathrm{m}$$

$$y_{\mathrm{r}} \geq \mathrm{y}_{\mathrm{ro}} \quad , \quad \mathrm{r} = 1, \dots, \mathrm{s}$$

$$\mu_j \geq 0 \quad , \quad j = 1, \dots, n \quad , j \neq o$$

where, $\propto = \sum_{r=1}^{s} y_{ro} - \sum_{i=1}^{m} x_{io}$ is a constant value. For a better understanding of the MAJ model, refer to Jahanshaloo et al (2004).

2.5.Information on PCA

The ranking method proposed by Zhu [5] is presented below:

$$d_{ri}^{j} = \frac{y_{rj}}{x_{ij}} \quad , \quad i = 1, \dots, m \quad , \quad r = 1, \dots, s \tag{5}$$

The higher d_{ir}^{j}, shows a better performance of DMU_j in terms of the r^{th}output and i^{th} input compared with other $DMUs$.

2.6.TOPSIS

Ideal point is called $DMU^{+}$and anti-ideal point is called DMU^{-}. Suppose $\{x_1^{+}, x_2^{+} \dots, x_m^{+}\}$ and $\{y_1^{+}, y_2^{+} \dots, y_s^{+}\}$ represent input and output vectors of DMU^{+}, and suppose$\{x_1^{-}, x_2^{-} \dots, x_m^{-}\}$ and $\{y_1^{-}, y_2^{-} \dots, y_s^{-}\}$ represent input and output vectors of DMU^{-}, respectively. Consider the following relations:

$$\begin{aligned} x_i^{+} &= min_{j=1}^{n} x_{ij} \quad , \quad i = 1, \dots, m \\ y_r^{+} &= max_{j=1}^{n} x_{rj} \quad , \quad r = 1, \dots, s \\ x_i^{-} &= max_{j=1}^{n} x_{ij} \quad , \quad i = 1, \dots, m \\ y_r^{-} &= min_{j=1}^{n} x_{rj} \quad , \quad r = 1, \dots, s \end{aligned} \tag{6}$$

$DMU_j's$ distance to the ideal point is calculated as follows:

$$D_j^{+} = \sqrt{\sum_{i=1}^{m} (x_{ij} - x_i^{+})^2 - \sum_{r=1}^{s} (y_{rj} - y_r^{+})^2} \tag{7}$$

$DMU_j's$ distance to the anti-ideal point is calculated as follows:

$$D_j^{-} = \sqrt{\sum_{i=1}^{m} (x_{ij} - x_i^{-})^2 - \sum_{r=1}^{s} (y_{rj} - y_r^{-})^{\ 2}} \tag{8}$$

Now, $DMU_j's$ general distance to the ideal point and the anti-ideal point has been provided here:

$$D_j = D_j^{-} / D_j^{-} + D_j^{+} \tag{9}$$

The larger value of D_j, represents better performance of DMU_j compared with other $DMUs$. For more information about the TOPSIS method, refer to [6].

3. A new Method for ranking Efficient DMUs Based on TOPSIS and Virtual DMUs

In this new method, first the ideal point and the anti-ideal point are obtained using relation (6), all the efficient DMUs are identified, and then obtained results are put in $E = \{1, \dots, e\}$. Continuing with all the efficient DMUs, we produced a virtual DMU. The acquired inputs and outputs of the virtual DMU are the average of the corresponding inputs and outputs of all the efficient $DMUs$, respectively. And it is called DMU_M (DMU_M is the center of gravity of all the efficient DMUs). After that, general distance of DMU_M to the ideal point and the anti-ideal point are obtained through applying relation (9), this distance is called D_M. In the next step, $DMU_q, q \in E$ is deleted from the efficient frontier and the new virtual DMU is created which is the average of the corresponding inputs and outputs of all the efficient DMUs, respectively, except DMU_q. This is called $DMU_{q'}$. After that, general distance of $DMU_{q'}$ to the ideal point and the anti-ideal point is obtained using relation (9), and it is called $D_{q'}$. Following the procedure, the true difference between D_M and $D_{q'}$ will be computed and named as $d_{q'}$. This method will be repeated for every $DMU_q, q \in E$. The distance $d_{q'}$ is the criteria for ranking efficient $DMUs$, the larger is $d_{q'}$ the better ranking DMU_q will have.

Algorithm

Step1. The ideal point and the anti-ideal point are obtained using relation (6).

Step2. We identify all the efficient $DMUs$ and put them in $\mathrm{E} = \{1, \dots, \mathrm{e}\}$.

Step3. Using all the efficient $DMUs$, we produce a virtual DMU, the inputs and outputs of the virtual DMU are the average of the corresponding inputs and outputs of all the efficient $DMUs$, respectively. It is called DMU_M (DMU_M is the center of gravity of all the efficient $DMUs$).

Step4. General distance of DMU_M to the ideal point and the anti-ideal point are obtained applying relation (9), which is called D_M.

Step5. $DMU_q, q \in E$ is deleted from the efficient frontier and the new virtual DMU is created from the average of the corresponding inputs and outputs of all the efficient $DMUs$, with the exception of DMU_q, which is called $DMU_{q'}$.

Step6. The general distance $DMU_{q'}$ to the ideal point and the anti-ideal point is obtained through relation (9), and is called $D_{q'}$.

Step7. Difference between D_M and $D_{q'}$ will be computed and named as $d_{q'}$.

Step8. This method will be repeated for every $DMU_q, q \in E$.

Step9. $d_{q'}$ is the criteria for ranking efficient $DMUs$, the larger is $d_{q'}$, the better ranking DMU_q will have.

4. Numerical Example

To illustrate the proposed ranking model, we consider an example with 28 DMUs which have 3 inputs and 3 outputs [4]. Consider Table 1:

Table1. Inputs and Outputs of 28 DMU$_s$

DMU	Input 1	Input 2	Input 3	Output 1	Output 2	Output 3
1	483.01	1,397,736	616,961	6,785,798	1,594,957	1,088,699
2	371.95	855,509	385,453	2,505,984	545,140	835,745
3	268.23	685,584	341,941	2,292,025	406,947	473,600
4	202.02	452,713	117,429	1,158,016	135,939	336,165
5	197.93	471,650	112,634	1,244,124	204,909	317,709
6	178.96	423,124	189,743	1,187,130	190,178	605,037
7	148.04	367,012	97,004	658,910	86,514	239,760
8	189.93	408,311	111,904	993,238	1,411,954	353,896
9	23.33	245,542	91,861	854,188	135,327	239,360
10	116.91	305,316	91,710	606,743	78,357	208,188
11	129.62	295,812	92,409	736,545	114,365	298,112
12	106.26	198,703	53,499	454,684	67,154	233,733
13	89.70	210,891	95,642	494,196	78,992	118,553
14	109.26	282,209	84,202	842,854	149,186	243,361
15	85.50	184,992	49,357	776,285	116,974	234,875
16	72.17	222,327	73,907	490,998	117,854	118,924
17	76.18	161,159	47,977	482,448	67,857	158,250
18	73.21	144,163	43,312	515,237	114,883	101,231
19	86.72	190,043	55,326	625,514	173,099	130,423
20	69.09	158,439	66,640	382,880	74,126	123,968
21	77.69	135,046	46,198	867,467	65,229	262,876
22	97.42	206,926	66,120	830,142	128,279	242,773
23	54.96	79,563	43,192	521,684	37,245	184,055
24	67.00	144,092	43,350	869,973	86,859	194,416
25	46.30	100,431	31,428	604,715	55,989	127,586
26	65.12	96,873	28,112	601,299	37,088	224,855
27	20.09	50,717	54,650	145,792	11,816	24,442
28	69.81	117,790	30,976	319,218	31,726	169,051

The ideal point and the anti-ideal point obtained by relation (6).

Table2. The Ideal Point and the Anti-Ideal Point

	Input 1	Input 2	Input 3	Output 1	Output 2	Output 3
Ideal	20.09	50717	28112	6785798	1594957	1088699
Anti-ideal	483.01	1397736	616961	145792	11816	24442

Using CCR model, unit 1, unit 2, unit 6, unit 8, unit 21, unit 23, unit 24, unit 25, unit 26 and unit 27 have become efficient. The efficient $DMUs$ are ranked by the new method. The results of ranking are shown in Table 3.

Table3. Ranking of the Efficient DMU$_s$

Efficient $DMUs$	$d_{q'}$	Rank
1	0.08925	1
2	-0.70655	10
6	0.07515	2
8	0.00731	3
21	0.00282	4
23	-0.00143	8
24	0.00269	5
25	-0.00058	7
26	-0.00039	6
27	-0.00668	9

Among the efficient DMU$_s$, DMU$_1$ receives 1stposition and DMU_2 receives 10th position. Table 4 shows the results of ranking with the proposed methods which are compared with other methods.

Table4. The Result of Comparison

$DMUs$	1	2	6	8	21	23	24	25	26	27
AP (input oriented)	infeasible	9	4	1	6	8	7	5	3	2
MAJ (input oriented)	infeasible	4	2	1	7	9	8	6	5	3
AP (output oriented)	6	1	5	7	2	4	3	infeasible	infeasible	infeasible
MAJ (output oriented)	6	3	5	7	2	4	1	infeasible	infeasible	infeasible
Norm L_1	1	8	3	2	7	10	9	5	6	4
Proposed method	1	10	2	3	4	8	5	7	6	9

Table 4 shows that both AP (input oriented) and MAJ (input oriented) models in unit 1 are infeasible, AP method (output oriented) is infeasible in units 25, 26, 27 and MAJ method (output oriented) is infeasible in units 25, 26, 27, but the proposed method is always feasible. Zhu’s method is applied to units 2, 6, 8, 21, 23, 24, 25, 26, 27 (Table 5).

Table5. The result of Applying Zhu Method

DMU	d_{11}	d_{12}	d_{13}	d_{21}	d_{22}	d_{23}	d_{31}	d_{32}	d_{33}
2	6737.42	2.9292	6.5013	1465.62	0.6372	1.4142	2246.92	0.9768	2.1682
6	6633.49	1062.68	3380.85	2.8056	0.4494	1.4299	6.2565	1.0022	3.1887
8	5229.49	7434.07	1863.29	2.4325	3.4580	0.8667	8.8758	12.617	3.1624
21	11165	839.60	3383.65	6.4234	0.4830	1.9465	18.777	1.4119	5.6902
23	9492.06	677.67	3348.8	6.5568	0.4681	2.3133	12.078	0.8623	4.2613
24	12984	1296.4	2901.7	6.0376	0.6028	2.0036	2901.7	1.3492	4.4847
25	13060	1209.2	2755.6	6.0211	0.5574	1.7815	19.241	1.2703	4.0596
26	9233.7	6.2070	21.389	569.5	0.3828	1.3192	3452.9	2.3211	7.9985
27	7256.9	2.8746	2.6677	588.15	0.2329	0.2162	1216.62	0.4819	0.4472

Largerd_{ir}^{j}, shows better performance of DMU_j in terms of the r^{th} output and i^{th} input compared with other $DMUs$. The unit 23 in the fifth ratio and the unit 24 in the seventh ratio are greater than unit 2, in norm L_1 method of ranking; it ranks unit 23 as 10, unit 24 as 9 and unit 2 as 8, but the proposed method ranks unit 23 as 8, unit 24 as 5 and unit 2 as 10. The unit 21 in the fifth and seventh ratios are greater than units 25, 26, 27 respectively, in norm L_1 ranking method, it ranks unit 21 as 7, unit 25as5, unit 26 as6 and unit27 as 4, but the proposed method ranks unit 21 as 4, unit 25 as 7, unit 26 as 6 and unit 27 as 9. The unit 24 in the seventh, fifth and eighth ratios is greater than units 25, 26 and 27 respectively, in normL_1ranking method, it ranks unit 24 as 9, unit 25 as 5, unit 26 is 6 and unit 27 as ranking 4, but the proposed method ranks unit 24 as 5, unit 25 is 7, unit 26 as6and unit 27 as 9. The unit 26 in the fifth and eighth ratios is greater than unit 25 and unit 27 respectively, in norm L_1 ranking method, it ranks unit 26 as 6, unit 25 as 5 and unit 27 as 4, but the proposed method ranks unit 26 as 6, unit 25 as 7 and unit 27 as 9. The unit 25 in the ninth ratio is greater than unit 27, in norm L_1 ranking method, it ranks unit 25 as 5, unit 27 as 4, but the proposed method ranks unit 25 as 7, and unit 27 as 9. The unit 6 in the fifth ratio is greater than unit 8, in norm L_1 ranking method, it ranks unit 6 as 3, unit 8 as 2, but the proposed method ranks unit 6 as 2, unit 8 as 3. Thus, it can be said that the proposed method, in comparison with normL_1 method, is the more accurate.

5. Conclusion

We introduced a new method for ranking efficient $DMUs$ based on TOPSIS and virtual $DMUs$. The comparison of the proposed method with other methods shows the proposed method is better than AP (input oriented), MAJ (input oriented), AP (output oriented), MAJ (output oriented) and norm L_1 methods. The proposed method is always feasible and simpler than other compared methods. As a forthcoming research, this study can be extended to the interval data.

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