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Classifications of Linking Activities Based on Their Inefficiencies in Network DEA

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Abstract

Network DEA models deal with measurements of relative efficiency of Decision-Making Units (DMUs) when the insight of their internal structures is available. In network models, sub-processes are connected by links or intermediate products. Links have the dual role of output from one division or sub-process and input to another one. Therefore, improving the efficiency score of one division by increasing its output may reduce the score of another division because of increasing its input. To address this conflict, in the present paper we proposed a new approach in Slack-Based Measure (SBM) framework which provides deeper insights regarding the sources of inefficiency. The proposed approach is a two-phase procedure in which Phase-I determine the role of intermediate measures by solving a linear program and partitions the intermediate measures into three groups of "input type", "output type" and "fixed-flows" and Phase-II measures the scores of the DMUs under evaluation. Providing a classification for intermediate products and account their excesses or shortfalls in efficiency calculation while the continuity of link flows between subunits are kept, are the advantages of the proposed approach.

Keywords: Network DEA, Intermediate product, SBM, Efficiency score.

1 | Introduction

Data Envelopment Analysis (DEA) was developed to measure the relative efficiency of operational units called "Decision Making Units (DMUs)" [1] that consume multiple inputs to produce multiple outputs. In its original settings, the operations and interrelations of the processes within the DMU are neglected and only the inputs to the DMU and outputs from it are considered. In the literature of DEA this approach is called black box and the associated model is called black box model.

In real world problems organizations have complex internal structures and many of them consist of several divisions or sub-units/processes that are linked together by products or services.

Regarding efficiency evaluation of multi-division organizations, in many situations, analysts considered the subunits as independent DMUs, and calculated their efficiencies separately. This approach is thus

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called separation approach and the associated model as separation model. In separation approach the links between two sub-units have the both role of input to one sub-unit and output from another one, hence the complex structure can be divided into sub-units or divisions and for each division some benchmarks can be found. Since links are treated as discretionary inputs or outputs, the separation model takes into account the inefficiency associated with the linking activities but does not keep the continuity of flows between subunits.

In the literature of DEA many researchers are interested in investigating the sources of inefficiency within the DMUs with complex structure and measuring divisional efficiencies as well as the overall efficiency in a unified framework. To accomplish this, researchers have developed Network DEA methodologies which are more sensitive in detecting inefficiencies than traditional DEA models.

Network DEA models were introduced for the first time to the literature of DEA by Färe and Grosskopf [2], [3]. After pioneering work of Färe and Grosskopf [2], a significant number of researchers and scholars have abandoned the black box perspective and started to look into the black box.

A network structure can be a simple two-stage process or a complex system with multiple divisions that are linked together with intermediate measures. Linking activities or intermediate measures are indispensable parts of Network DEA models. Since they play the both roles of input to one division and output from another one, their existence is the major problem in measuring overall and divisional efficiencies of a network. Improving the efficiency of one division by increasing its outputs may reduce the efficiency of another division due to the increased amount of inputs. Similarly, raising efficiency score of one division by reducing the amount of its input may reduce the efficiency of the other division due to producing output. Since standard DEA models do not resolve this conflict, they are not a good choice for assessing the efficiency of DMUs with network structure and therefore, many scholars and researchers have proposed their own solutions.

For instance, Yong et al. [4] defined the overall efficiency score as a product of stages' efficiency scores. They applied the input-oriented VRS model to measure the efficiency of the first stage and the output-oriented VRS model to measure the efficiency of the second stage.

Kao and Hwang [5] also defined the overall efficiency score of the two-stage structure as the geometric mean of stage efficiencies. Liang et al. [6] also use such multiplicative efficiency decomposition in their study. They use the concepts of the Stackelberg game (or leader-follower) and the centralized or cooperative game to evaluate the overall efficiency score.

Lewis and Sexton [7] proposed a Network DEA model with two-stage structure for measuring the efficiency of Major League Baseball (MLB) teams. Their methodology provides efficiency scores for each stage and overall efficiency. In 2004, they extended their model to a multi stage structure and in 2013 they propose an un-oriented two-stage DEA methodology to measure efficiency of MLB teams during the 2009 season [8], [9].

All Network DEA models mentioned above apply the CCR [1] or the BCC [10] models as the basic DEA methodology and the production possibility set. In other words, they apply the radial measure of efficiency that rely on the assumption that DMU's efficiency score depends on its proportional distance to the efficiency frontier. However, in real words problems some inputs and outputs are substitutional and do not change proportionally. Non-radial models have the advantage of measuring efficiencies in the case that inputs and outputs change non-proportionally. One of the most popular non-radial models in the literature of DEA is Slack-Based Measure (SBM) model [11], [12]. Non-radial SBM models deal with slacks directly and do not consider the assumption of changing inputs and outputs proportionally.

Tone and Tsutsui [13] develop a slacks-based Network DEA model by using the production possibility Sets. In their study intermediate measures are called links. They considered the component efficiency as a function of slack variables and the overall efficiency as a weighted average of the component efficiencies. In their study the component weights are determined exogenously to represent the importance of the components. They proposed two possible cases for linking activities, called fixed link and free link. In both cases the continuity of link flows between components is kept. Fukuyama and Weber [14] proposed a measure for efficiency called network directional SBM. They normalized values of the slack variables by user defined coefficients.

Paradi et al. [15] proposed a modified Slacks-Based Measure (SBM) for evaluating efficiency of a DMU with two-stage structure to aggregate the obtained efficiency scores from the stages and generate a composite performance index for each unit.

Lozano [16] proposed an SBM model for measuring efficiency of networks. In their proposed approach the target inputs, outputs and intermediate products of each process may be larger or smaller than their observed values. By relaxing the constraints for both the fixed-link and the free-link cases they improved the discriminating power of their model.

Shamsijamkhaneh et al. [17] proposed an approach which categorizes the intermediate measures into either input or output type endogenously, and keeps the continuity of link flows between divisions. Based on their approach they proposed two models to study on direct and indirect effect of inefficiency arising from intermediate measure in efficiency measurement.

In this paper we propose a two-phase procedure in SBM framework to measure the overall and divisional efficiencies of the DMUs under consideration. The proposed procedure accounts the excesses or shortfalls of intermediate measures into the objective function. The major contribution of this paper is to address the conflict caused by the dual role of intermediate measures and incorporate their excesses and shortfalls in efficiency measurement. The main novelty in our proposed approach lies in the more flexible manner in which we categorize the intermediate measures by allowing them to have a small violation from their observed values and ignoring their small excesses and shortfalls.

Kord et al. [18] proposed a new Network DEA model to evaluate the sustainability of agricultural performance in the cities of Sistan-Baluchestan Province of Iran in the presence of stochastic data. They considered two stages for agricultural practices: the environmental stage (planting and maintaining) and the economic stage (harvesting), which use shared resources.

Abdali et al. [19] proposed a multiplier two-stage network that simulate the internal structure of network systems in parallel-series structure in the presence of non-discretionary inputs and shared discretionary inputs between sub-DMUs.

Pereira et al. [20] proposed a Network DEA model to measure the performance of countries in struggle against health crises like SARS-CoV-2. They considered the countries as dmus with a general series structure with five stages, population, contagion, triage, hospitalization, and intensive care unit admission. They suggested an output orientation model for a social perspective, and an input orientation model for a financial perspective.

Hamzah et al. [21] and Mariano et al. [22] evaluated the performance of heath system in Brazil and Malaysia, respectively in fighting against COVID-19.

Zhang et al. [23] stated all models in Network DEA literature generally assume radial or non-radial point of views to declare internal structures of DMUs and no study used the intermediate point of view to construct a Network DEA model. To fill the literature gap, they proposed a model which combines the intermediate approach with Network DEA and develop a new approach to measure the efficiency of a network.

Roudabr et al. [24] proposed a novel model on the basis of Network DEA to determine the most suitable benchmarks for DMUs and SUB-DMUs. In their proposed model, input and output values consider to have nonlinear values. They applied their proposed model to determine the benchmark for cement factories listed on the Tehran Stock Exchange.

Zhu et al. [25] introduced a model based on Mixed Integer Linear Program (MILP) which finds the most efficient targets on the extended production possibility set in DEA.

Yang et al. [26] applied Network DEA on water systems by proposing a dynamic interactive network SBM model. They considered the water systems as two water subsystems, Water Use Subsystem (WUS) and a Wastewater Treatment Subsystem (WTS) and evaluated the relative efficiency of regional industrial water systems in China.

Li et al. [27] applied a four-stage Network DEA based on SBM to measure the total factor waste gas treatment efficiency in steel and iron industries in China. They entered the data of 65 Chinese company during the years 2005-2014 in their model.

The rest of this paper is structured as follows; Section 2 presents some preliminaries and notation. Section 3 presents our proposed procedure to address the issue regarding the dual role of intermediate measures. To verify our proposed procedure, we provide a numerical example in Section 4 and compare the procedure with some existing approaches in Network DEA. Finally, Section 5 concludes the paper.

2 | Preliminaries

In this section, we will review some fundamental backgrounds required in this paper.

Suppose that there are a set of n DMUs indexed by consisting of K divisions that Division K (Divk) consumes inputs and produces outputs. Let and, respectively, be input vector to and output vector from Divk. Intermediate products from Divk to Divh are also denoted by where is the number of intermediate measures from Divk to Divh and denotes the set of links.

Tone and Tsutsui [13] propose a Network DEA (NDEA) model based on the Weighted Slacks-Based Measure (WSBM) approach to measure the overall and divisional efficiencies of the network. Their model presented as follows:

$$\rho = \min \sum_{k=1}^{K} w_k \left[\frac{1 - \frac{1}{m_k} \left(\sum_{i=1}^{m_k} \frac{s_{pi}^{k-}}{x_{ip}} \right)}{1 + \frac{1}{r_k} \left(\sum_{r=1}^{r_k} \frac{s_{rp}^{k+}}{y_{rp}} \right)} \right], \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j^k x_{ij}^k + s_{ip}^{k-} = x_{ip}^k, \quad (k = 1,\dots,K), (i = 1,\dots,m_k), \tag{2}$$

$$\sum_{j=1}^{n} \lambda_j^k y_{rj}^k - s_{rp}^{k+} = y_{rp}^k, \quad (k = 1,\dots,K), (r = 1,\dots,r_k), \tag{3}$$

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} \quad (\text{for all } (k,h)), (d = 1,\dots,l_{(k,h)}), \tag{4}$$

$$\sum_{j=1}^{n} \lambda_j^h z_{dj}^{(k,h)} = z_{dp}^{(k,h)} \quad (\text{for all } (k,h)), (d = 1,\dots,l_{(k,h)}), \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j^k = 1 \quad (\text{for all } k), \tag{6}$$

$$\lambda_j^k \geq 0 \quad \text{for all} j,k), s_{rp}^{k+} \geq 0 \quad \text{for all } r,k), s_{ip}^{k-} \geq 0 \quad \text{for all } r,k), s_{dp}^{k,h)-} \geq 0 \ (\text{for all } d, \text{ for all } (k,h)), s_{dp}^{k,h)+} \geq 0 \ (\text{for all } d, \ \text{for all } (k,h)), \tag{7}$$

where is the intensity weight corresponding to Divk of DMUj, and is also the relative weight of Divk which is determined exogenously by decision maker to represent its importance and $\sum_{k=1}^{K} w^k = 1$, $w^k \geq 0 \ (for\ all\ \ k)$.

It should be noted that the model presented above computes the non-oriented overall efficiency of DMUp under the assumption of Variable Returns-to-Scale (VRS) for production. Removing *Constraint (6)* changes the assumption of VRS to the Constant Returns-to-Scale (CRS) for production. Tone and Tsutsui [13] proposed the input and output-oriented case of their model by minimizing the numerator and maximizing the denominator of the objective *Function (1)*, respectively.

In the model presented above, linking *Constraints (4)* and *Constraint (5)* are kept unchanged and fixed, and the intermediate products are beyond the control of DMUs. Tone and Tsutsui [13] called this case as "fixed" link value case.

Substituting *Constraint (4)* and *Constraint (5)* by *Constraint (8)*, they introduced another possible case for linking activities called "free" link value case in which the linking activities can be freely determined.

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} = \sum_{j=1}^{n} \lambda_j^h z_{dj}^{(k,h)} \quad \text{for all } (k,h)(d = 1,\ldots,l_{(k,h)}). \tag{8}$$

Note that in both cases the continuity of link flows between divisions are kept.

In the case that intermediate measures are categorized into either input type or output type exogenously by decision maker, Tone and Tsutsui [13] incorporate the input excesses and output shortfalls by setting the linking *Constraint (9)* and *Constraint (10)* and modifying the objective *Functions (1)* to *(11)*.

As output to Divh

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j^k z_j^{(k,h)} - s_{dp}^{(k,h)+} = z_{dp}^{(k,h)}, \\ &\sum_{j=1}^{n} \lambda_j^h z_{dj}^{(k,h)} = \sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)}. \end{aligned} \tag{9}$$

As input to Divh

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j^h z_j^{(k,h)} + s_o^{(k,h)-} = z_o^{(k,h)}, \\ &\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} = \sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)}. \end{aligned} \tag{10}$$

$$\eta_p^* = \min \sum_{k=1}^{K} w_k \left[\frac{1 - \frac{1}{m_k + l_k} \left(\sum_{i=1}^{m_k} \frac{s_{pi}^{k-}}{x_{ip}} + \sum_{h \in F_k}^{h_k} \frac{s_{hp}^{(h,k)-}}{z_{hp}^{(k,f)}} \right)}{1 + \frac{1}{r_k + t_k} \left(\sum_{r=1}^{r_k} \frac{s_{rp}^{k+}}{y_{rp}} + \sum_{t=1}^{t_k} \frac{s_{tp}^{(k,f)+}}{z_{tp}^{(k,f)}} \right)} \right], \tag{11}$$

where $\sum_{k=1}^{K} w_k = 1,\ w_k \geq 0\ \left(for\ all\ k\right)$. t_k is the number of those intermediate products that are considered as output from Divk and l_k is the number of those intermediate products that are considered as input to Divk.

There are many situations in which the intermediate measures cannot be categorized into input or output type by the decision maker. For instance, consider the buyer-seller supply chain presented in Liang et al. [28] the supplier's revenue is an output from the seller, and seller wants to increase it while also an input to the buyer and buyer interested to decrease it. Therefore, there is always a conflict between buyer and seller and minimizing the total supply chain cost or maximizing the total supply chain revenue (profit).

In the next section we propose a two-phase procedure in SBM framework which classifies the intermediate measures into three groups of "input type", "output type" and "fixed-flows" and identifies the potential improvements regarding linking activities.

3 | Proposed Procedure

In this section we introduce a Two-Phase procedure to measure the relative efficiencies of DMUs with network structure. This procedure addresses the issue regarding dual role of intermediate measures in efficiency measurement and incorporate inefficiencies associated with intermediate measures in efficiency measurement. Phase-I is a linear program model which partitions intermediate measures into three groups of "input type", "output type" and "fixed-flows". According to the results obtained from Phase-I, in Phase-II we use SBM model to determine the slack of each input, output and intermediate measure and we incorporate these slacks in measuring efficiencies of the DMUs under consideration.

Phase-I

As we discussed earlier, in Phase-I we propose a linear program to partition the intermediate products into input type, output type and fixed- flows. We use the linear program in *Eqs. (12)* to *(18)* for Phase-I.

$$\beta = \max \sum_{k=1}^{K} w_k \left(\sum_{i=1}^{m_k} \frac{s_{pi}^{k-}}{x_{ip}} + \sum_{r=1}^{r_k} \frac{s_{rp}^{k+}}{y_{rp}} + \sum_{d=1}^{l_{(k,h)}} \frac{s_{dp}^{(k,h)+}}{z_{dp}} + \sum_{d=1}^{l_{(k,h)}} \frac{s_{dp}^{(k,h)-}}{z_{dp}} \right). \tag{12}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j^k x_{ij}^k = x_{ip}^k - s_{ip}^{k-}, \quad (k=1,\dots,K),(i=1,\dots,m_k), \tag{13}$$

$$\sum_{j=1}^{n} \lambda_j^k y_{rj}^k = y_{rp}^k + s_{rp}^{k+}, \quad (k=1,\dots,K),(r=1,\dots,r_k), \tag{14}$$

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} - s_{dp}^{(k,h)-} + s_{dp}^{(k,h)+} \quad (\text{for all } (k,h)),(d=1,\dots,l_{(k,h)}), \tag{15}$$

$$\sum_{j=1}^{n} (\lambda_j^h - \lambda_j^k) z_{dj}^{(k,h)} = 0 \quad (\text{for all } (k,h)),(d=1,\dots,l_{(k,h)}), \tag{16}$$

$$\sum_{j=1}^{n} \lambda_j^k = 1 \quad (\text{for all } k), \tag{17}$$

$$\begin{gathered} \lambda_j^k \geq 0\ (\text{for all } j,k), s_{rp}^{k+} \geq 0 \quad (\text{for all } r,k), s_{ip}^{k-} \geq 0 \quad (\text{for all } r,k), s_{dp}^{(k,h)-} \geq 0 \\ (\text{for all } d, \text{for all } (k,h)),\ s_{dp}^{(k,h)+} \geq 0 \quad (\text{for all } d,\ \text{for all } (k,h)). \end{gathered} \tag{18}$$

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In Phase-I, the objective function maximizes the total improvement ratios of each input, output and intermediate measure of DMU_p. The set of left-side of *Constraint (13)* to *Constraint (17)* is the efficient frontier with respect to DMU_p. The right-side of *Eq. (13)* is the improved ith input at Divk located on the frontier.

The right-side of *Constraint (14)* is the expanded rth output at Divk located on the frontier.

The right-side of *Constraint (15)* is the improved dth link between Divk and Divh located on the frontier.

It should be noted the slack vectors related to $s_{dp}^{(k,h)-}$ and $s_{dp}^{(k,h)+}$ could not have been used to form a basis simultaneously, since they are linearly dependent; hence at least one of these variables is nonbasic and its optimal value is zero.

In particular, the following conditions must hold at optimality:

I. $s_{dp}^{(k,h)-}$ is basic and $s_{dp}^{(k,h)+}$ is nonbasic ($s_{dp}^{(k,h)+}=0$). In this case the corresponding intermediate measure is considered as input to Divh. Let $D^{(k,h)-}$ denote the set of links that are considered as output from Divk.

II. $s_{dp}^{(k,h)+}$ is basic and $s_{dp}^{(k,h)-}$ is nonbasic ($s_{dp}^{(k,h)-}=0$). In this case the corresponding intermediate measure is considered as output from *Divk* and we denote the set of these links by $D^{(k,h)+}$.

III. $s_{dp}^{*(k,h)+}$ and $s_{dp}^{*(k,h)-}$ are both nonbasic ($s_{dp}^{*(k,h)-}=s_{dp}^{*(k,h)+}=0$). In this case the corresponding link is fixed and kept unchanged. We denote the set of these intermediate measures by $D_{fixed}^{(k,h)}$.

Refer to explanations above it is easy to conclude that $D^{(k,h)-}\cup D^{(k,h)+}\cup D_{fixed}^{(k,h)}=D^{(k,h)}$ and $D^{(k,h)-}\cap D^{(k,h)+}\cap D_{fixed}^{(k,h)}=\varphi$.

Phase-II

According to the results obtained from Phase-I, in Phase-II we aim to measure the relative efficiencies of DMUs by incorporating the inefficiency associated with intermediate measures in efficiency measurement. We use the linear program *Eqs. (19)* to *(27)* for Phase-II that is the input-oriented SBM under VRS assumption for evaluating efficiency score. The non-oriented or output-oriented models can also be utilized for Phase-II.

$$\varphi_p^* = \min \sum_{k=1}^{K} w_k \left[1 - \frac{1}{m_k + \left|D^{(f,k)-}\right|} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{k-}}{x_{ip}} + \sum_{d=1}^{\left|D^{(f,k)-}\right|} \frac{s_{dp}^{(f,k)-}}{z_{dp}^{(f,k)}} \right) \right], \tag{19}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j^k x_{ij}^k = x_{ip}^k - s_{ip}^{k-}, \quad (k=1,\dots,K),(i=1,\dots,m_k), \tag{20}$$

$$\sum_{j=1}^{n} \lambda_j^k y_{rj}^k = y_{rp}^k + s_{rp}^{k+}, \quad (k=1,\dots,K),(r=1,\dots,r_k), \tag{21}$$

$$\sum_{j=1}^{n} \lambda_j^h z_{dj}^{(k,h)} = z_{dp}^{(k,h)} - s_{dp}^{(k,h)-} \quad \text{for all } (k,h), d \in D^{(k,h)-}, \tag{22}$$

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} + s_{dp}^{(k,h)+} \quad \text{for all } (k,h), d \in D^{(k,h)+}, \tag{23}$$

$$\sum_{j=1}^{n}\lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} \quad \text{for all } (k,h), d \in D_{\text{fixed}}^{(k,h)}, \tag{24}$$

$$\sum_{j=1}^{n}(\lambda_j^h - \lambda_j^k) z_{dj}^{(k,h)} = 0 \quad (\text{for all } (k,h)), (d = 1,\dots,l_{(k,h)}), \tag{25}$$

$$\sum_{j=1}^{n}\lambda_j^k = 1 \quad (\text{for all } k), \tag{26}$$

$$\begin{aligned}&\lambda_j^k \geq 0 \quad (\text{for all } j,k),\ s_{rp}^{k+} \geq 0 \quad (\text{for all } r,k),\ s_{ip}^{k-} \geq 0 \quad (\text{for all } r,k), s_{dp}^{(k,h)-} \geq 0 \\ &(\text{for all } d,\ \text{for all } (k,h)),\ s_{dp}^{(k,h)+} \geq 0 \quad (\text{for all } d,\ \text{for all } (k,h)).\end{aligned} \tag{27}$$

The objective *Function (19)* states the minimum mean proportional reduction rate of inputs or input mix inefficiencies.

Theorem 1. For the proposed procedure, every division has at least one divisionally efficient DMU.

Proof: As we noticed earlier in Phase-I the intermediate measures are partitioned into input type, output type and fixed- flows. Therefore, the proposed procedure in Phase-II can be reduced to the Separation model with non-discretionary inputs and outputs corresponding to the fixed-flows. Hence, we can solve this case separately division by division. Therefore, every division has at least one efficient DMU in the division.

4 | Numerical Example

In this section to verify our proposed procedure, we present a numerical example and compare the results with NSBM model in free-link case.

Consider the numerical example given in *Table 1* where we have seven DMUs consist of 3 divisions and each division has a single exogenous input. There are two final outputs which correspond to Div2 and Div3 and there are two intermediate products that one links Div1 to Div2 and the other links Div2 to Div3. *Fig. 1* displays the network structure of the DMUs under consideration.

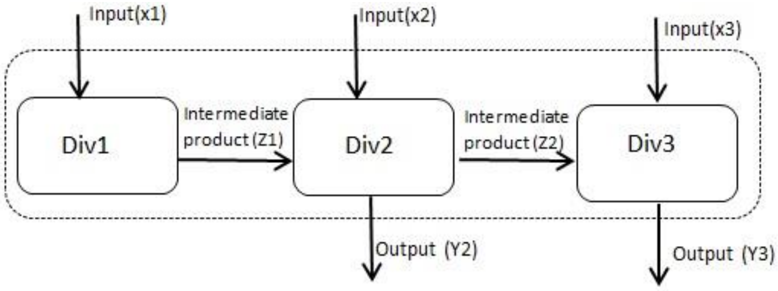

Fig. 1. Network structure of the DMUs.

Table 1. Data for numerical example.

DMU	Div1 Input1 (x_1)	Div2 Input2 (x_2)	Output2 (y_2)	Div3 Input3 (x_3)	Output3 (y_3)	link Link12 (z_1)	Link23 (z_2)
DMU1	1.5	1.2	1.5	2.5	1.1	1	1.5
DMU2	1.25	1.5	2	1.1	2.5	0.4	0.8
DMU3	1.45	0.75	1.1	3	3	2.8	2
DMU4	1.25	1	1	1.2	1	4.5	4
DMU5	1.3	0.25	1.2	1.9	0.8	2.9	2.2
DMU6	2	1	1.25	1.2	2.2	2.1	2.5
DMU7	1.1	0.75	1.4	1.8	1.9	1.5	0.9

In this section we consider the numbers 0.4, 0.2 and 0.4 as the weights to Div1, Div2 and Div3, respectively in models and we utilize input-oriented SBM under the VRS assumption for efficiency evaluation in all models.

4.1 | The Results Obtained by Proposed Procedure

In this subsection we use the proposed procedure to obtain the overall and divisional scores of the DMUs with data exhibited in *Table 1*.

As we discussed earlier, Phase-I partitions the intermediate measures into three groups of input type, output type and fixed flows. *Table 2* represents the results of Phase-I.

Table 2. The results obtained by solving Phase-I.

DMU	Obj	s^{*1-}	s^{*2-}	s^{*3-}	s^{*2+}	s^{*3+}	$s^{*(1,2)+}$	$s^{*(1,2)-}$	Type of z1	$s^{*(2,3)-}$	$s^{*(2,3)+}$	Type of z2
DMU1	2.104	1.222	0.109	1.218	0.173	0.182	0	0	Input	0	0.645	Fixed
DMU2	0.911	1.139	0	0	0	0	0	0	Fixed	0	0	Fixed
DMU3	0.778	0.831	0	0	0	0	0	0.572	Input	0	0	Fixed
DMU4	3.167	0.444	0.750	0.144	0.2	0.936	0	1.6	Input	1.8	0	Input
DMU5	2.245	0.494	0	0.844	0	1.136	0	0	Fixed	0	0	Fixed
DMU6	1.767	1.632	0	0.065	0	0.194	0	0.775	Input	1.1	0	Input
DMU7	1.246	0	0	0	0	0	0.4	0	Output	0	0.882	Output

Intermediate measure Z1 is considered as input to Div2 in DMU3, DMU4 and DMU6 and intermediate measure Z2 is considered as input to Div3 in DMU1, DMU4 and DMU6. Intermediate measure Z1 is kept unchanged in DMU1, DMU2 and DMU5 and intermediate measure Z2 is kept unchanged in DMU2, DMU3 and DMU5. Both Z1 and Z2 are considered as outputs from Div1 and Div2, respectively, in DMU7.

With the partitions of the links exhibited in *Table 2*, we are ready to employ Phase-II. Divisional and overall scores for the DMUs are listed in *Table 3*.

Table 3. Divisional and overall scores of the DMUs obtained from proposed procedure.

DMU	Overall	Div1	Div2	Div3
DMU1	0.302	0.185	0.794	0.173
DMU2	0.636	0.089	1.000	1.000
DMU3	0.601	0.353	0.229	1.000
DMU4	0.192	0.044	0.397	0.238
DMU5	0.602	0.620	1.000	0.385
DMU6	0.355	0.057	0.575	0.544
DMU7	0.694	0.379	1.000	0.485

4.2 | Comparisons of Scores between Proposed Procedure and NSBM

In this subsection we compare our proposed approach with NSBM in free link case proposed by Tone and Tsutsui [13]. *Table 6* exhibits the results obtained by NSBM model in free-link case.

Table 4. The results of NSBM model in free-link case.

	Overall Efficiency	Divisional Efficiency			Initial Input Slacks		
DMU	Free-link	Div1(0.4)	Div2(0.2)	Div3(0.4)	s^{*1-}	s^{*2-}	s^{*3-}
DMU1	0.332	0.156	0.938	0.206	1.417	0.075	1.984
DMU2	0.636	0.089	1.000	1.000	1.139	0.000	0.000
DMU3	0.665	0.504	0.314	1.000	0.719	0.515	0.000
DMU4	0.318	0.044	0.750	0.375	1.194	0.250	0.750
DMU5	0.602	0.620	1.000	0.385	0.494	0.000	1.169
DMU6	0.526	0.079	0.860	0.807	1.886	0.046	0.232
DMU7	0.694	0.379	1.000	0.485	0.683	0.000	0.926

Fig. 2 illustrates the comparison of the results obtained by proposed procedure and NSBM model in free-link case. It can bee seen that how the inefficiencies associated with linking activities exert influence over

divisional and overall efficiency of each DMU. Considering the feasible regions of the proposed procedure and NSBM free-link case, it can be easily concluded that the feasible region of NSBM free link case contains the feasible region of Phase-II.

The objective of phase-II can take values smaller than or equal to those of NSBM model, hence it can be certainly said that the scores obtained by the proposed approach are not definitely higher than those of NSBM.

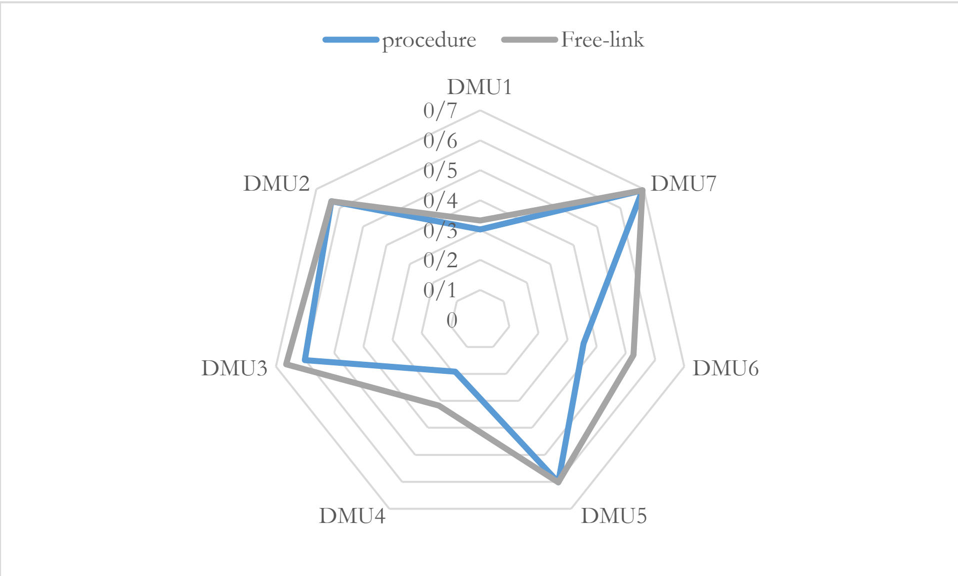

Fig. 2. Comparisons of overall efficiency scores.

Comparing the results obtained by proposed procedure and free-link case shows that DMU2, DMU5 and DMU7 have the same efficiency score in both models. This means that intermediate measures in these dmus do not have any excesses. Comparing the scores obtained by both models for DMU4 shows that there are significant inefficiencies due to intermediate measures which are considered as inputs to divisions 2 and 3 by the model and both model evaluated the lowest efficiency score for DMU4. According to both models DMU7 have the highest score among the other DMUs. The different scores for DMU1 indicates the inefficiency due to intermediate measure which is detected by the model as input to division2.

5 | Conclusion

To address the potential conflict caused by the dual role of intermediate measures and incorporate the inefficiencies associated with intermediate measures in efficiency measurement in this paper we proposed a new procedure in SBM frame work. The proposed approach has the advantage of optimizing the system structure and the slack values simultaneously and partitions the intermediate measures. Phase-I determines the role of intermediate measures by solving a linear programming and partitions the intermediate measures into three groups of input type, output type and fixed flows. The objective of Phase-I maximizes the total improvement ratios of each input, output and intermediate measures of the DMU under consideration. With the partitions of the links in Phase-I, we employ Phase-II to measure the scores of the DMUs under consideration. We demonstrated that for the proposed procedure in Phase-II, every division has at least one divisionally efficient DMU. For further research we can suggest the following issues.

Extending the procedure to the situation in which some input/output data are fuzzy numbers. Another possible line of research is to extend the procedure to the dynamic network models.

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