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A Numerical Solution for the Fractional Ideal Equation of Thermoelectric Coolers

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Abstract

One of the fields studied in the science of heat physics is the thermoelectric phenomenon. This phenomenon is in fact the interaction between the current of electricity and the thermal properties of a system. In simpler terms, it is a phenomenon in which the direct conversion of a temperature difference to voltage occurs. In this paper, we introduced a method based on the finite difference technique for solving a fractional differential equation in the field of thermal physics which describes the thermoelectric phenomena, numerically. For this purpose, we used fractional order derivatives with the definitions of Caputo, finite differences with the second order central finite-difference approach, and the first order central finite-difference. By using this method, we translate the desired differential equation to a system of nonlinear differential equations which can be solved. Finally, some numerical are used to demonstrate the effective and accuracy of the scheme. The obtained numerical results show that our proposed method is highly accurate.

Keywords: Thermal physics, Thermoelectric phenomenon, Finite difference, Fractional order.

1 | Introduction

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One of the ways to study the behavior of physical phenomena is to model them using the equations governing these phenomena by mathematical tools, in which differential equations are undoubtedly one of the most powerful tools in this regard [1]-[3]. Differential equations as a branch of mathematics are powerful tools in many scientific fields such as geology, chemistry, physics, engineering and other sciences [4].

One method of solving differential equations is the finite difference method [5]. The first application of finite difference methods was published in the second decade of the 20^{th} century by Richardson on fluid dynamics. In recent years, many researchers have developed numerical models to solve problems in various fields of science using this method.

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One of the fields studied in the science of heat physics is the thermoelectric phenomenon [6]. This phenomenon is in fact the interaction between the current of electricity and the thermal properties of a system. In simpler terms, it is a phenomenon in which the direct conversion of a temperature difference to voltage occurs. One of the most famous classical equations in this phenomenon is the heat flux equation (heat flow density vector) and is as follows:

$$\vec{q} = -k\vec{\nabla}T + ST\vec{j}, \tag{1}$$

where S is Seebeck coefficient, T is the absolute temperature, j is the current density and k is the thermal conductivity. In fact, *Eq. (1)* expresses the net cooling power in terms of heat flux vector [7]. Ones can write the steady-state of heat diffusion equation as follows [7]:

$$\nabla.\left(k\vec{\nabla}T\right) + j^2\varphi - T\frac{dS}{dT}\vec{j}.\vec{\nabla}T = 0, \tag{2}$$

where the φ is the electrical resistivity and as we can see it is a nonlinear differential equation. In *Eq. (2)* the first, second and the third terms are the thermal conduction, the Joule heating and the Thomson heat, respectively [7]. It is usually assumed that the Thomson effect does not exist or can be ignored [8]-[10].

The fractional calculus raised in the 18th century. It is a branch of mathematics that orders of derivatives and integrals are arbitrary. In fact, it is a natural extension of classical mathematics. Really, this matter has recently become an increasingly important topic in the literature of many sciences such as applied mathematics, engineering, and so on. It has attracted the notice of many scientists in different fields of sciences [2], [11], [12]. Different methods have been used to solve this type of equation [13]-[15].

The *Eq. (2)* has been transformed into an ordinary nonlinear differential equation by using dimensionless variables as follows [7]:

$$y'' - abyy' + a(b-1)y' + c = 0. \quad y(0) = 0, \quad y(1) = 1 \tag{3}$$

Eq. (2) of the fractional order is as follows:

$$y^{(\alpha)} - abyy' + a(b-1)y' + c = 0, y(0) = 0. \quad y(1) = 1, \quad 1 < \alpha \le 2 \tag{4}$$

In this paper, we intend to solve the *Eq. (4)* by the finite difference method and then observe the effect of the fractional order of the derivative of the equation through the change in the order of the derivative. Finally, we compare the obtained solutions by drawing their graphs.

2 | Definitions, Basic Concepts and Formulas

In this section, we present some basic concepts, definitions, formulas, block pulse functions and the fractional calculations.

Definition 1. When $z(t) \in L_1[0, b]$ we will have the fractional derivative in Caputo sense as follows [4]:

$$D^{\alpha}z(t) = \begin{cases} I^{n-\alpha}D_t^n z(t), \ n-1 < \alpha \le n, \ n \in \mathbb{N}, t > 0 \\ \dfrac{d^n}{dt^n}z(t). \qquad \alpha = n \end{cases} \tag{5}$$

For a constant value, the derivative by means of Caputo is 0, and we have:

$$D_x^{\alpha}t^n = \begin{cases} 0, & n \in \mathbb{N}, n < \lceil\alpha\rceil \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}, & n \in \mathbb{N}, n > \lceil\alpha\rceil \end{cases} \tag{6}$$

where $\lceil\alpha\rceil$ is the smallest integer number larger from α [4].

Definition 2 ([4]). The Riemann-Liouville fractional derivative of order α where with respect to the variable t and with the starting point $t = a$ is

$$D^{\alpha}z(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dt^n}\displaystyle\int_a^t (t-\tau)^{n-1-\alpha}z(\tau)d\tau, & 0\le n-1\le\alpha<n\\ \dfrac{d^n}{dt^n}z(t), & \alpha=n\in\mathbb{N}\end{cases} \quad (7)$$

By means of the Riemann-Liouville sense, the fractional integral of order α is defined as [4]

$$I^{\alpha}\big(z(t)\big)={}_aD_t^{-\alpha}z(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}z(\tau)d\tau. \qquad \alpha>0 \quad (8)$$

The relation between the Caputo operator and Riemann-Liouville is as follows [4]

$${}_aD_t^{\alpha}I^{\alpha}z(t)=z(t). \quad (9)$$

$$I^{\alpha}\,{}_aD^{\alpha}z(t)=z(t)-\sum_{k=0}^{n-1} z^{(k)}(a^+)\frac{(t-a)^k}{k!}. \qquad t>0 \quad (10)$$

Lemma 1 ([4]). Let $\alpha,\beta\ge 0, d_1,d_2\in\mathbb{R}$ and $k(t),g(t)\in L_1[0,p]$, and then

$$D^{\alpha}\big(d_1g(t)+d_2k(t)\big)=d_1D^{\alpha}g(t)+d_2D^{\alpha}k(t). \quad (11)$$

$$I^{\alpha}I^{\beta}g(t)=I^{\beta}I^{\alpha}g(t), I^{\alpha}I^{\beta}g(t)=I^{\alpha+\beta}g(t). \quad (12)$$

The basic concepts of finite differences are described in detail in various books [5]. Finite differences were introduced by Brook Taylor in 1715 and have also been studied as abstract self-standing mathematical objects in works by George Boole in 1860, Milne-Thomson in 1933, and Karoly Jordan in 1939. Finite differences trace their origins back to one of Jost Burgi's algorithms in 1592 and work by others including Isaac Newton [16]. A finite difference is a mathematical expression of the form $f(x+b)-f(x+a)$. If a finite difference is divided by $b-a$, one gets a difference quotient. The approximation of derivatives by finite differences plays a central role in finite difference methods for the numerical solution of differential equations, especially boundary value problems.

The difference operator, commonly denoted Δ is the operator that maps a function f to the function $\Delta[f]$ defined by $\Delta[f](x)=f(x+1)-f(x)$. A difference equation is a functional equation that involves the finite difference operator in the same way as a differential equation involves derivatives. There are many similarities between difference equations and differential equations, especially in the solving methods. Certain recurrence relations can be written as difference equations by replacing iteration notation with finite differences. In numerical analysis, finite differences are widely used for approximating derivatives, and the term "finite difference" is often used as an abbreviation of "finite difference approximation of derivatives" [17]-[19]. Finite difference approximations are finite difference quotients in the terminology employed above. Three basic types are commonly considered: forward, backward, and central finite differences [17]-[19].

A forward difference, denoted $\Delta_h[f]$ of a function f is a function defined as $\Delta_h[f](x)=f(x+h)-f(x)$. Depending on the application, the spacing h may be variable or constant. When omitted, h is taken to be 1; that is $\Delta_1[f](x)=\Delta[f](x)=f(x+1)-f(x)$.

A backward difference uses the function values at x and $x-h$, instead of the values at $x+h$ and x as $\nabla_h[f](x)=f(x)-f(x-h)$.

Finally, the central difference is given by $\delta_h[f](x)=f\left(x+\frac{h}{2}\right)-f(x-\frac{h}{2})$. So here are just a few definitions needed in this regard.

Shariffar et al. | Int. J. Res. Ind. Eng. 11(3) (2022) 258-272

Definition 3. The finite-difference grid can be defined as follows: the solutions region in the $x-y$ space is defined by a rectangular grid with dimensions Δx and Δy along the axes of x and y, respectively. *Fig. 1* shows this grid.

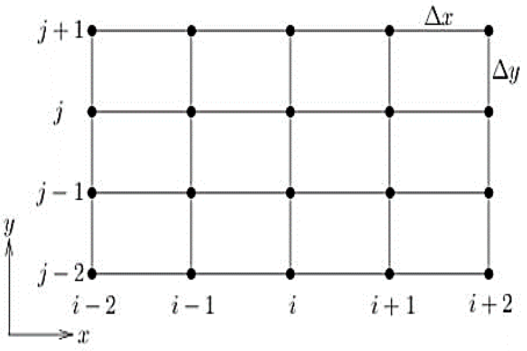

Fig. 1. Finite difference grid.

In this method, the function and its derivatives are approximated by finite differences. In this paper, we use two types of finite difference approximations, which we define below.

Definition 4. The approximation of the first-order derivative by centered difference is defined as follows:

$$y'(t_n) \simeq \frac{y_{n+1}-y_{n-1}}{2h}, \tag{13}$$

where y_{n+1} and y_{n-1} are the function values at points $(n+1)$ and $(n-1)$ of the network, respectively.

Definition 5. The approximation of the second-order derivative by centered difference is defined as follows:

$$y''(t_n) \simeq \frac{y_{n+1}-2y_n+y_{n-1}}{h^2}, \tag{14}$$

where y_{n+1} ,y_n and y_{n-1} are the function values at points $(n+1)$, n and $(n-1)$ of the network, respectively.

3 | Numerical Method

In this section, we first solve the *Eq. (4)* with the finite difference method. Then, with the same method, we solve the mentioned equation with fractional derivative in Caputo sense.

Case1. Suppose $\alpha = 2$, so the *Eq. (4)* is as follows:

$$y'' - abyy' + a(b-1)y' + c = \circ. \quad y(\circ) = \circ, \qquad y(1) = 1 \tag{15}$$

Let $t_n = nh$, $n = \circ, 1, \ldots, N$, $t_\circ = \circ$, $t_N = 1$, $y_\circ = \circ$, $y_N = 1$, then we approximate the derivatives in equation with finite differences as follows:

$$\begin{cases} y'(t_n) \approx \dfrac{y_{n+1}-y_{n-1}}{2h}, \\ y''(t_n) \approx \dfrac{y_{n+1}-2y_n+y_{n-1}}{h^2}. \end{cases} \tag{16}$$

By replacing these approximations in the mentioned equation, we have

$$\frac{y_{n+1}-2y_n+y_{n-1}}{h^2} - aby_n\frac{y_{n+1}-y_{n-1}}{2h} + a(b-1)\frac{y_{n+1}-y_{n-1}}{2h} + c = \circ. \tag{17}$$

Now for $n = 1$ we get

$$\frac{y_2 - 2y_1}{h^2} - aby_1\frac{y_2}{2h} + a(b-1)\frac{y_2}{2h} + c = \circ. \tag{18}$$

And for $2 \le n \le N-2$ we have

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} - aby_n\frac{y_{n+1} - y_{n-1}}{2h} + a(b-1)\frac{y_{n+1} - y_{n-1}}{2h} + c = \circ. \tag{19}$$

Finally, for $n = N-1$ we will have

$$\frac{1 - 2y_{N-1} + y_{N-2}}{h^2} - aby_{N-1}\frac{1 - y_{N-2}}{2h} + a(b-1)\frac{1 - y_{N-2}}{2h} + c = \circ. \tag{20}$$

As we can see, at any step we can obtain unknowns by using known values.

Case 2. Suppose $1 < \alpha \le 2$, so the *Eq. (4)* is as follows:

$$y^{(\alpha)} - abyy' + a(b-1)y' + c = \circ, y(\circ) = \circ, y(1) = 1. \tag{21}$$

By using definition of fractional derivative *Eq. (7)* we have

$$y^{(\alpha)}(t) = \frac{1}{\Gamma(2-\alpha)}\int_{\circ}^{t}(t-s)^{1-\alpha}y''(s)ds. \tag{22}$$

Now, we suppose $t_n = nh, n = \circ, 1, \dots, N, t_\circ = \circ, t_N = 1, y_\circ = \circ, y_N = 1$, then we calculate the fractional derivative in *Eq. (22)* for $t_{n+1/2}$ so we have

$$y^{(\alpha)}\left(t_{n+1/2}\right) = \frac{1}{\Gamma(2-\alpha)}\int_{\circ}^{t_{n+1/2}}\left(t_{n+1/2} - s\right)^{1-\alpha}y''(s)ds =$$

$$\frac{1}{\Gamma(2-\alpha)}\left(\begin{array}{c}\sum_{j=1}^{n}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\left(t_{n+\frac{1}{2}} - s\right)^{1-\alpha}y''(s)ds + \\ \int_{\circ}^{t_{1/2}}\left(t_{n+1/2} - s\right)^{1-\alpha}y''(s)ds\end{array}\right). \tag{23}$$

By replacing the $y''(s) = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}$ in *Eq. (23)* we have

$$y^{(\alpha)}\left(t_{n+1/2}\right) \simeq \frac{1}{\Gamma(2-\alpha)}\left(\begin{array}{c}\sum_{j=1}^{n}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\left(t_{n+\frac{1}{2}} - s\right)^{1-\alpha}\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}ds \\ + \int_{t_{-\frac{1}{2}}}^{t_{\frac{1}{2}}}\left(t_{n+\frac{1}{2}} - s\right)^{1-\alpha}\frac{y_1 - 2y_\circ + y_{-1}}{h^2}ds\end{array}\right). \tag{24}$$

By applying the initials and simplifying we obtain

$$y^{(\alpha)}\left(t_{n+1/2}\right) = \frac{1}{h^2\Gamma(2-\alpha)}\left(\begin{array}{c}\sum_{j=1}^{n}\left(y_{j+1} - 2y_j + y_{j-1}\right)\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\left(t_{n+\frac{1}{2}} - s\right)^{1-\alpha}ds + \\ y_1\int_{t_{-1/2}}^{t_{1/2}}\left(t_{n+1/2} - s\right)^{1-\alpha}ds\end{array}\right). \tag{25}$$

Now in *Eq. (25)* we assume the existing integral as follows:

$$d_{n,j,\alpha} = \int_{t_{j-1/2}}^{t_{j+1/2}}\left(t_{n+1/2} - s\right)^{1-\alpha}ds. \tag{26}$$

Shariffar et al.| Int. J. Res. Ind. Eng. 11(3) (2022) 258-272

Now by solving it we will have

$$d_{n,j,\alpha} = \int_{t_{j-1/2}}^{t_{j+1/2}} \left(t_{n+1/2} - s\right)^{1-\alpha} ds = \frac{1}{2-\alpha}\left((t_{n+1/2} - t_{j+1/2})^{2-\alpha} - (t_{n+1/2} - t_{j-1/2})^{2-\alpha}\right). \tag{27}$$

With simplification *Eq. (27)* we can write

$$d_{n,j,\alpha} = \frac{h^{2-\alpha}}{2-\alpha}\left((n-j+1)^{2-\alpha} - (n-j)^{2-\alpha}\right). \tag{28}$$

And for $j = \circ$ we have

$$d_{n,\circ,\alpha} = \frac{h^{2-\alpha}}{2-\alpha}\left((n+1)^{2-\alpha} - (n)^{2-\alpha}\right). \tag{29}$$

By replacing *Eq. (28)* in *Eq. (25)* we get

$$y^{(\alpha)}\left(t_{n+1/2}\right) = \frac{1}{h^2\Gamma(2-\alpha)}\left(\sum_{j=1}^{n}\left(y_{j+1} - 2y_j + y_{j-1}\right)d_{n,j,\alpha} + y_1 d_{n,\circ,\alpha}\right). \tag{30}$$

In *Eq. (30)* on the right-hand side, we rewrite the first sentence according to its limits, so we have

$$\begin{aligned}\sum_{j=1}^{n}\left(y_{j+1} - 2y_j + y_{j-1}\right)d_{n,j,\alpha} &= d_{n,1,\alpha}y_\circ + \left(-2d_{n,1,\alpha} + d_{n,2,\alpha}\right)y_1 \\ &\quad + \left(d_{n,1,\alpha} - 2d_{n,2,\alpha} + d_{n,3,\alpha}\right)y_2 \\ &+\left(d_{n,2,\alpha} - 2d_{n,3,\alpha} + d_{n,4,\alpha}\right)y_3 + \cdots \\ &+\left(d_{n,n-3,\alpha} - 2d_{n,n-2,\alpha} + d_{n,n-1,\alpha}\right)y_{n-2} \\ &+\left(d_{n,n-2,\alpha} - 2d_{n,n-1,\alpha} + d_{n,n,\alpha}\right)y_{n-1} + \left(d_{n,n-1,\alpha} - 2d_{n,n,\alpha}\right)y_n + d_{n,n,\alpha}y_{n+1}.\end{aligned} \tag{31}$$

With a little calculation and by simplifying *Eq. (31)* we have

$$\begin{aligned}\sum_{j=1}^{n}\left(y_{j+1} - 2y_j + y_{j-1}\right)d_{n,j,\alpha} &= d_{n,1,\alpha}y_\circ + \left(-2d_{n,1,\alpha} + d_{n,2,\alpha}\right)y_1 + \\ &\sum_{j=2}^{n-1}\left(d_{n,j-1,\alpha} - 2d_{n,j,\alpha} + d_{n,j+1,\alpha}\right)y_j + \left(d_{n,n-1,\alpha} - 2d_{n,n,\alpha}\right)y_n + d_{n,n,\alpha}y_{n+1}.\end{aligned} \tag{32}$$

By replacing *Eq. (32)* in *Eq. (30)* we get

$$y^{(\alpha)}\left(t_{n+1/2}\right) \approx \frac{1}{h^2\Gamma(2-\alpha)}\left(\begin{aligned}&\sum_{j=1}^{n-1}\left(d_{n,j-1,\alpha} - 2d_{n,j,\alpha} + d_{n,j+1,\alpha}\right)y_j + \\ &\left(d_{n,n-1,\alpha} - 2d_{n,n,\alpha}\right)y_n + d_{n,n,\alpha}y_{n+1}\end{aligned}\right). \tag{33}$$

By replacing $y'\left(t_{n+1/2}\right) \approx \frac{y_{n+1}-y_n}{h}$, $y\left(t_{n+1/2}\right) \approx \frac{y_{n+1}-y_n}{2}$ and *Eq. (33)* in *Eq. (21)* we obtain

$$\begin{aligned}&\frac{1}{h^2\Gamma(2-\alpha)}\left(\sum_{j=1}^{n-1}\left(d_{n,j-1,\alpha} - 2d_{n,j,\alpha} + d_{n,j+1,\alpha}\right)y_j + \left(d_{n,n-1,\alpha} - 2d_{n,n,\alpha}\right)y_n \right. \\ &\qquad\left. + d_{n,n,\alpha}y_{n+1}\right) \\ &-ab\frac{(y_{n+1}-y_n)^2}{2h} + a(b-1)\frac{y_{n+1}-y_n}{h} + c = \circ.\end{aligned} \tag{34}$$

Now for $n = 1$ we have

$$\frac{1}{h^2\Gamma(2-\alpha)}\left(\left(d_{1,\circ,\alpha}-2d_{1,1,\alpha}\right)y_1+d_{1,1,\alpha}y_2\right)-ab\frac{(y_2-y_1)^2}{2h}+a(b-1)\frac{y_2-y_1}{h}+c=\circ. \tag{35}$$

In the following for $2 \le n \le N-2$ we have

$$\frac{1}{h^2\Gamma(2-\alpha)}\left(\sum_{j=1}^{n-1}\left(d_{n,j-1,\alpha}-2d_{n,j,\alpha}+d_{n,j+1,\alpha}\right)y_j+\left(d_{n,n-1,\alpha}-2d_{n,n,\alpha}\right)y_n\right.$$
$$\left.+d_{n,n,\alpha}y_{n+1}\right)-ab\frac{(y_{n+1}-y_n)^2}{2h}+a(b-1)\frac{y_{n+1}-y_n}{h}+c=\circ. \tag{36}$$

Finally, for $n = N-1$ we will have as follows:

$$\frac{1}{h^2\Gamma(2-\alpha)}\left(\sum_{j=1}^{N-2}\left(d_{N-1,j-1,\alpha}-2d_{N-1,j,\alpha}+d_{N-1,j+1,\alpha}\right)y_j+\left(d_{N-1,N-2,\alpha}-\right.\right.$$
$$\left.\left.2d_{N-1,N-1,\alpha}\right)y_{N-1}+d_{N-1,N-1,\alpha}\right)-ab\frac{(1-y_{N-1})^2}{2h}+a(b-1)\frac{1-y_{N-1}}{h}+c=\circ. \tag{37}$$

Again, as we can see, at any step we can obtain unknowns by using known values.

4 | Stability

Let $\|.\|$be the usual L^2 norm. Assume that $\tilde{y}_n$ and $\hat{y}_n$ are exact and approximation solutions of *Eq. (17)*, respectively. Hence,

$$\frac{\tilde{y}_{n+1}-2\tilde{y}_n+\tilde{y}_{n-1}}{h^2}-ab\tilde{y}_n\frac{\tilde{y}_{n+1}-\tilde{y}_{n-1}}{2h}+a(b-1)\frac{\tilde{y}_{n+1}-\tilde{y}_{n-1}}{h^2}+c=0. \tag{38}$$

$$\frac{\hat{y}_{n+1}-2\hat{y}_n+\hat{y}_{n-1}}{h^2}-ab\hat{y}_n\frac{\hat{y}_{n+1}-\hat{y}_{n-1}}{2h}+a(b-1)\frac{\hat{y}_{n+1}-\hat{y}_{n-1}}{2h}+c=0. \tag{39}$$

We define the roundoff error $e_n = \tilde{y}_n - \hat{y}_n$. From *Eq.* (17) the following relationship can be obtained

$$\frac{e_{n+1}-2e_n+e_{n-1}}{h^2}-\frac{ab}{2h}\left(\tilde{y}_n(\tilde{y}_{n+1}-\tilde{y}_{n-1})-\hat{y}_n(\hat{y}_{n+1}-\hat{y}_{n-1})\right)+a(b-1)\frac{e_{n+1}-e_{n-1}}{2h}=0,$$

Or

$$\left(1+\frac{1}{2}ha(b-1)\right)e_{n+1}=2e_n-\left(1-\frac{1}{2}ha(b-1)\right)e_{n-1}+\frac{1}{2}abh\left(\tilde{y}_n(\tilde{y}_{n+1}-\tilde{y}_{n-1})-\right.$$
$$\left.\hat{y}_n(\hat{y}_{n+1}-\hat{y}_{n-1})\right). \tag{40}$$

Thus, we have

$$\left|1+\frac{1}{2}ha(b-1)\right|\|e_{n+1}\|\le 2\|e_n\|+\left|1-\frac{1}{2}ha(b-1)\right|\|e_{n-1}\|+\frac{1}{2}|ab|h\left\|\left(\tilde{y}_n(\tilde{y}_{n+1}-\right.\right.$$
$$\left.\left.\tilde{y}_{n-1})-\hat{y}_n(\hat{y}_{n+1}-\hat{y}_{n-1})\right)\right\|. \tag{41}$$

Theorem 1 ([20]). Suppose f maps a convex open set $E \subseteq \mathbb{R}^m$, f is differentiable in E, and there is a real nember M such that $\|f''\| \le M$ for every $x \in E$. Then

$$f(b)-f(a)\le M|b-a| \text{ for all } a\in E, b\in E.$$

Also, in this paper we have

$$|||e_n|||=\max\{\|e_{n-1}\|,\|e_n\|,\|e_{n+1}\|\}. \tag{42}$$

From *Eqs. (41), (42)* and theorem 1 we obtain

$$\left|1+\frac{1}{2}ha(b-1)\right|\|e_{n+1}\|\le 2\|e_n\|+\left|1-\frac{1}{2}ha(b-1)\right|\|e_{n-1}\|+\frac{1}{2}|ab|h|||e_n|||, \text{ or} \tag{43}$$

Shariffar et al.| Int. J. Res. Ind. Eng. 11(3) (2022) 258-272

$$\left|1+\frac{1}{2}ha(b-1)\right| \|e_{n+1}\| \leq \left(2+\left|1-\frac{1}{2}ha(b-1)\right|+\frac{1}{2}|ab|h\right) |||e_n|||,$$

Thus,

$$\|e_{n+1}\| \leq \frac{1}{\left|1+\frac{1}{2}ha(b-1)\right|}\left(2+\left|1-\frac{1}{2}ha(b-1)\right|+\frac{1}{2}|ab|h\right) |||e_n|||.$$

According to *Eq. (42)* we know that there is a real number K such that $|||.||| \leq K\|.\|$,

Therefore,

$$\|e_{n+1}\| \leq \frac{K_n}{\left|1+\frac{1}{2}ha(b-1)\right|}\left(2+\left|1-\frac{1}{2}ha(b-1)\right|+\frac{1}{2}|ab|h\right) \|e_n\|. \quad (44)$$

So,

$$\|e_{n+1}\| \leq C_n\|e_n\|. \quad (45)$$

Where $C_n = \frac{K_n}{\left|1+\frac{1}{2}ha(b-1)\right|}\left(2+\left|1-\frac{1}{2}ha(b-1)\right|+\frac{1}{2}|ab|h\right)$.

From *Eq. (45)* we have $\|e_{n+1}\| \leq D\|e_0\|$,

Where

$$D = C_n C_{n-1} \dots C_0. \quad (46)$$

Therefore, according to *Eq. (46)* the scheme in (17) is stable in L^2.

5 | Illustrative Examples

As we know, ones use numerical methods are used to approximate the solution of equations generally do not have an analytical and exact solution. *Eq. (21)* is also one of these equations, so in the following examples, first, we solve some cases with the method described in the previous section. Then we present its accuracy by drawing the graph of the solution and comparing it with the graph of the exact solution (by selecting the existing coefficients with specific values). Then we present the error graph resulting from using the method along with the error table. Then, by changing the coefficient values and also changing the order of the derivative from integer to fractional, we analyze the solutions based on the graphs.

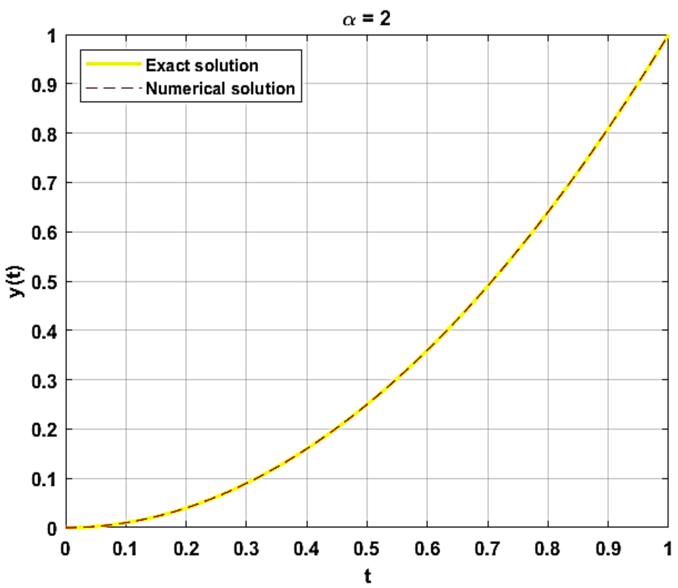

Fig. 2. Comparison between numerical and exact solution in example 1.

Example 1. In *Eq. (21)*, we set $\alpha = 2$, $a = 0$, $b = 0$ and $c = -2$ so, in this case we have $y'' - 2 = 0$. As we know, the exact solution of this equation is $y(t) = t^2$. *Fig. 2* shows a comparison of the graph of the solution obtained from the numerical method and the exact solution of the equation.

As can be seen from *Fig. 2*, the method has acceptable accuracy. The *Table 1* shows the error of method at $t \in [0,1]$ and $h = 0.01$ Also, *Fig. 3* presents the graph of the error.

Table 1. Values of error for example 1.

Time	Exact Solution	Numerical Solution	Error
0	0.00000000000000000	0.00000000000000000	0
0.1	0.01000000000000000	0.00999999999999987	1.31839E-16
0.2	0.04000000000000000	0.03999999999999980	2.49800E-16
0.3	0.09000000000000000	0.08999999999999960	3.88578E-16
0.4	0.16000000000000000	0.16000000000000000	4.99600E-16
0.5	0.25000000000000000	0.25000000000000000	4.44089E-16
0.6	0.36000000000000000	0.36000000000000000	1.66533E-16
0.7	0.49000000000000000	0.49000000000000000	3.33067E-16
0.8	0.64000000000000000	0.64000000000000000	2.22045E-16
0.9	0.81000000000000000	0.81000000000000000	3.33067E-16
1	1.00000000000000000	1.00000000000000000	0

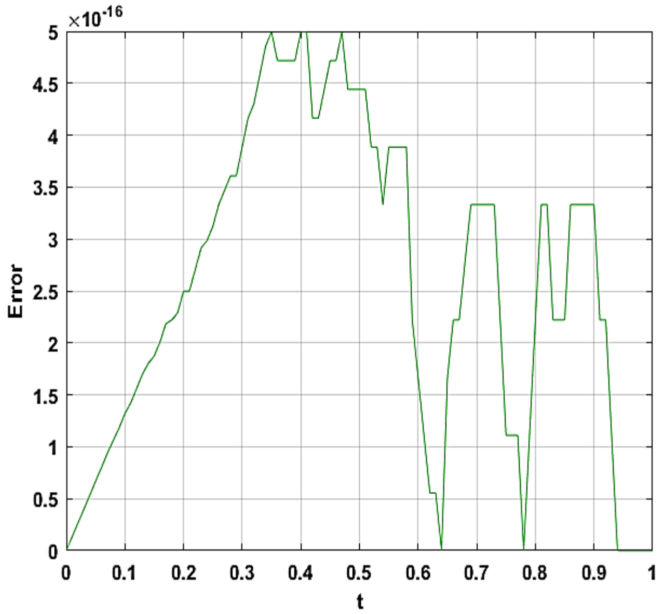

Fig. 3. The graph of error in example 1.

Now, by changing the order of the derivative from 2 to 1.8 and1.6, respectively, we draw the graphs of the solutions in *Fig. 4*. According to this Figure, the changes made in the graph of the solutions due to the change in the order of the derivatives from integer to fractional are quite obvious. From a mathematical point of view, the slope of the graph decreases with decreasing order, and in a part of the interval ($t < 0.3$), negative solutions are obtained. The convergence of the solutions is also easily visible.

Shariffar et al.| Int. J. Res. Ind. Eng. 11(3) (2022) 258-272

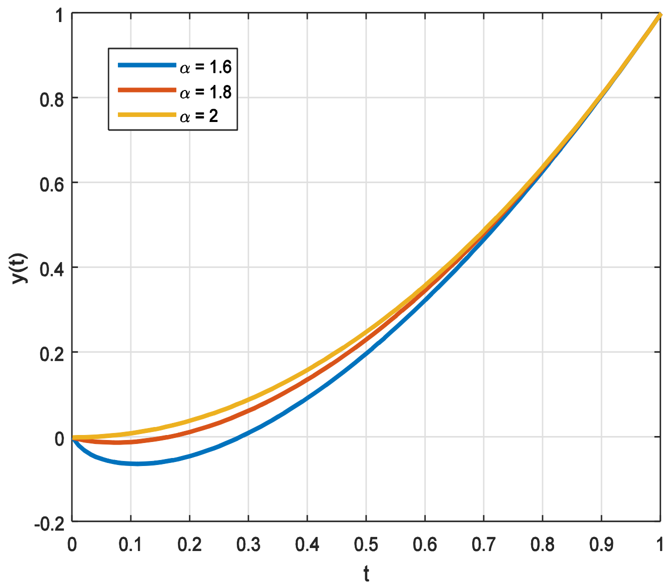

Fig. 4. Comparison of the graph of the solutions based on the order of derivatives.

From a physical point of view, the phase graph of a problem, in fact, is the drawing of solutions against the derivatives of the equation. So, it is of great importance. *Fig. 5* shows the phase diagram of this problem. As can be seen from this Figure, the critical points of the graphs are got away from the origin by decreasing the order of the derivative.

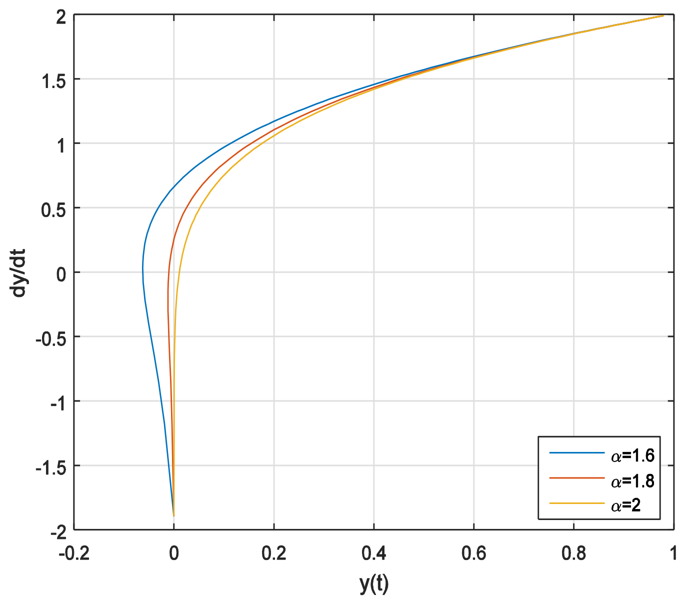

Fig. 5. The phase graph of example 1.

Example 2. In *Eq. (21)*, we set $\alpha = 2$, $a = 2$, $b = \circ$ and $c = 1$ so, in this case we know that the exact solution of this equation is $y(t) = \frac{1}{(2e^2-2)}(e^{2t} - 1) + \frac{t}{2}$. *Fig.* 6 shows a comparison of the graph of the solution obtained from the numerical method and the exact solution of the equation. Again *Fig. 6* shows, the method has good and acceptable accuracy.

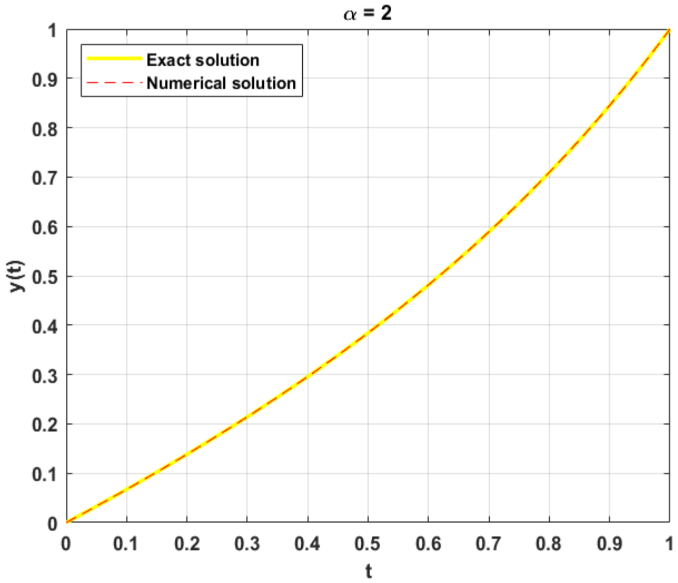

Fig. 6. Comparison between numerical and exact solution in example 2.

The *Table 2* shows the error of method at $t \in [0,1]$ and $h = 0.01$. Also, *Fig. 7* presents the graph of the error.

Table 2. Values of error for example 2.

Time	Exact Solution	Numerical Solution	Error
0	0	0	0
0.1	0.067326719	0.06732602	6.98703931E-07
0.2	0.138489621	0.13848821	1.41101358E-06
0.3	0.214338048	0.21433594	2.10870258E-06
0.4	0.295909389	0.295906638	2.75037861E-06
0.5	0.384470711	0.384467434	3.27703694E-06
0.6	0.481569616	0.48156601	3.60629057E-06
0.7	0.589096348	0.589092724	3.62490871E-06
0.8	0.709359658	0.709356479	3.17919737E-06
0.9	0.845179459	0.845177396	2.06263244E-06
1	1	1	0

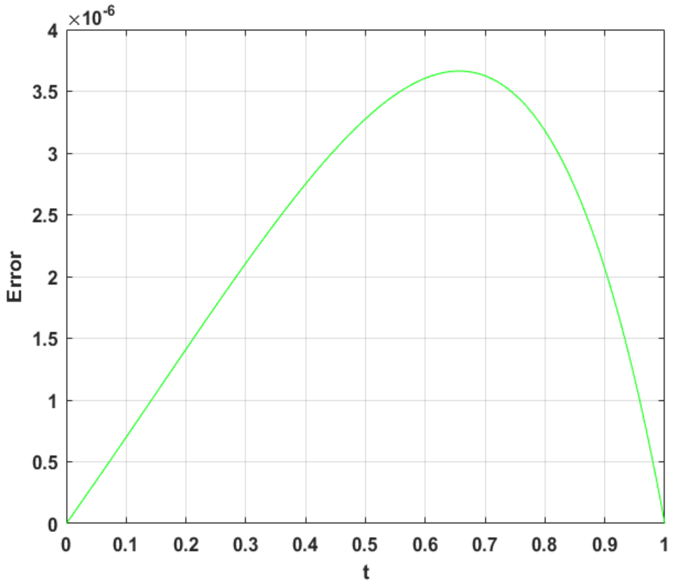

Fig. 7. The graph of error in example 2.

Shariffar et al.| Int. J. Res. Ind. Eng. 11(3) (2022) 258-272

In the following, in *Example 2*, we change the derivative order from 2 to 1.9 and 1.7, respectively, and then plot the solutions in *Fig. 8*. It is clear that before the graph of the solutions converges, it can be said that with decreasing the order of the derivative, the slope of the related graphs increases.

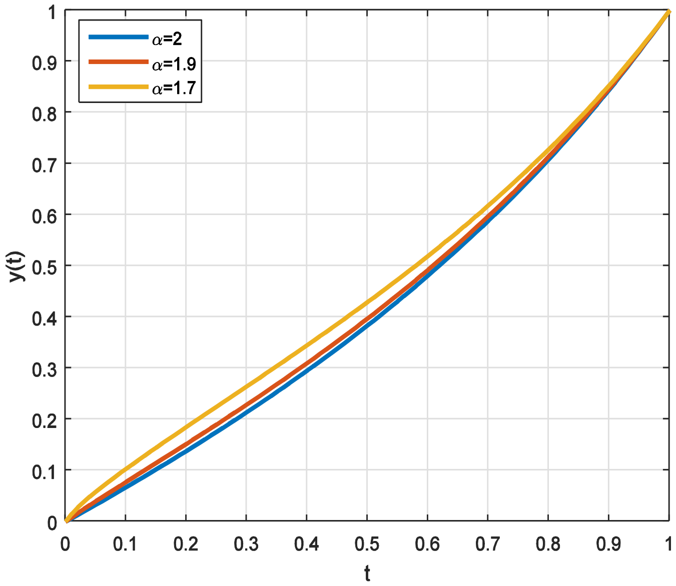

Fig. 8. Comparison of the graph of the solutions based on the order of derivatives.

The phase graph of *Example 2* is shown in *Fig. 9*. As can be seen from this Figure, the slope of the graphs decreases with decreasing order of the derivative, or in other words, the speed of the displacement or transfer reaction decreases

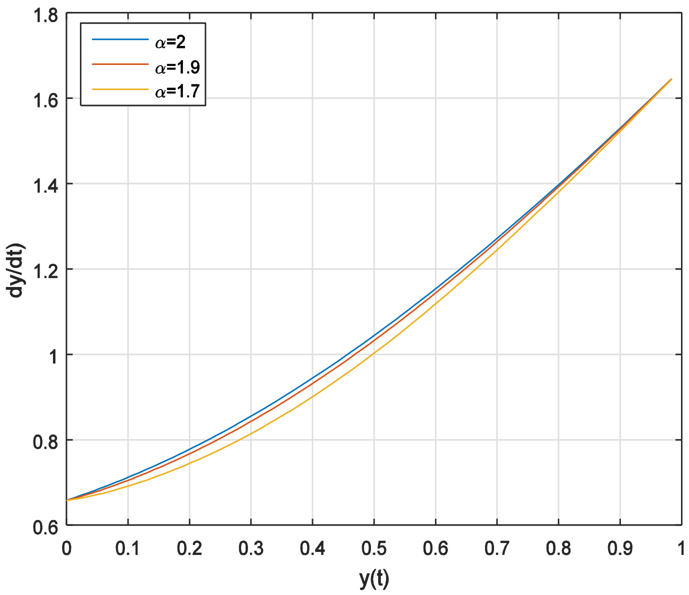

Fig. 9. The phase graph of example 2.

Example 3. In *Eq. (21)* we assume $a = 1$, $\boldsymbol{b} = \mathbf{2}$, $\boldsymbol{c} = \mathbf{3}$ and $\boldsymbol{\alpha} = \mathbf{2}$. With these assumptions, the mentioned equation does not have an exact solution, so we approximate the solution for the assumption coefficients. *Fig. 10* shows the result.

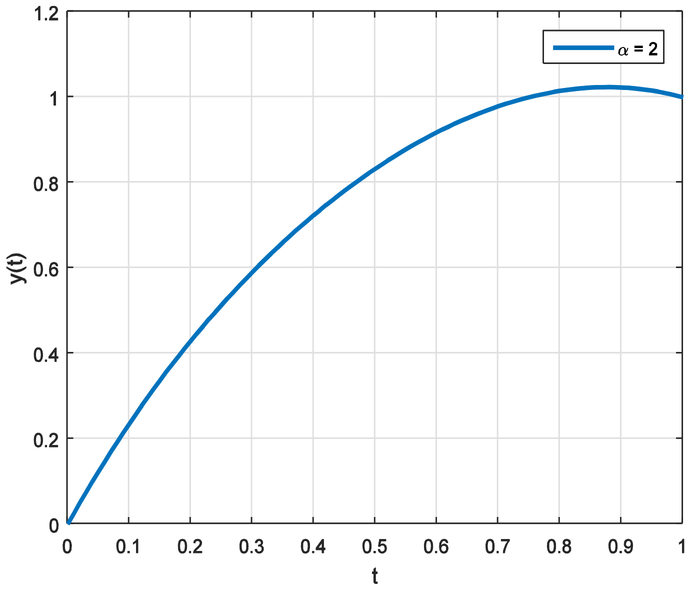

Fig. 10. The approximated solutions in example 3.

By keeping the coefficients of the equation constant, we change the order of the derivative in the equation from 2 to 1.9 and 1.7, respectively. *Fig. 11* shows the graph of the solutions of the equation with the mentioned derivatives. As can be seen from this Figure, the slope of the graph is shown by a significant increase with decreasing order. At the beginning of the interval, the rate of increase of this slope is very high, but as we move towards the end of the interval, this speed decreases, and in addition, at the end of the mentioned interval, the graphs converge around $t = 1$. The phase graph of this example is shown in *Fig. 12*.

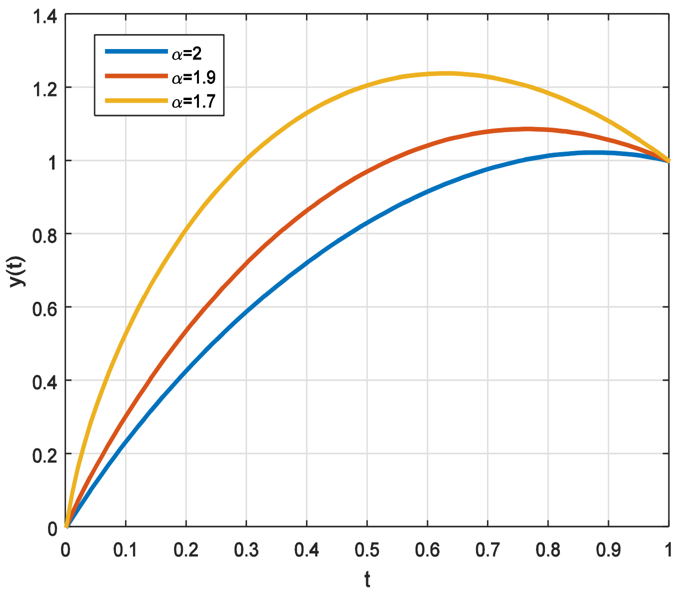

Fig. 11. Comparison of the graph of the solutions based on the order of derivatives.

Shariffar et al.| Int. J. Res. Ind. Eng. 11(3) (2022) 258-272

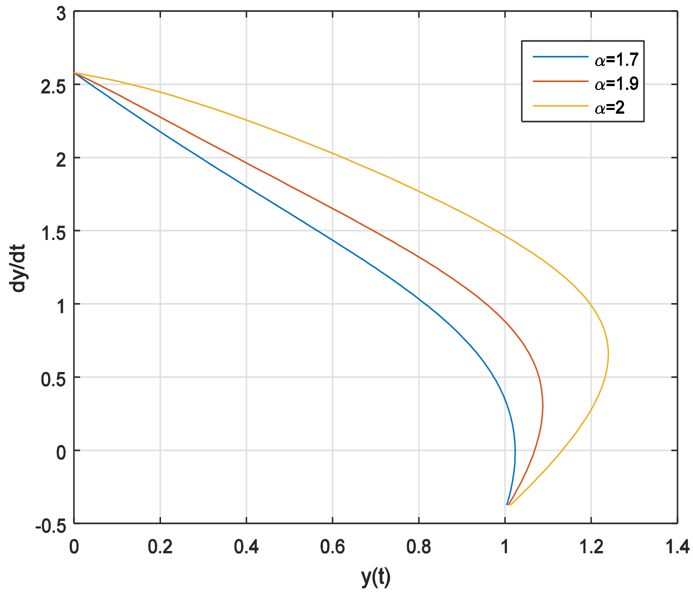

Fig. 12. The phase graph of example 3.

6 | Conclusion

In the presented work, we studied the Fractional Ideal Equation of Thermoelectric Coolers by finite difference method, numerically. First, we described the method in this paper. This method translated the fractional differential equation to a nonlinear algebraic equations system and then we solved it. We showed the efficiency and accuracy of the method with a few examples and by plotting the obtained solutions. In each of these examples, we presented the error graph and the table of error values and also examined the graph of the changes in the order of the equation from the integer to the fractional, as well as the phase graph of the equations. Then we solved the main equation and again examined the results by drawing the changes in the order of the equation from the integer to the fractional. The Phase graphs were also examined for the main equation of the fractional order. The obtained results show some behaviors of the system that can be seen only in the case where the order of equations is fractional.

Conflicts of Interest

All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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Shariffar et al.| Int. J. Res. Ind. Eng. 11(3) (2022) 258-272