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[0,1] Truncated Fréchet-Weibull and Fréchet Distributions

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A B S T R A C T

In this paper, we introduce a new family of continuous distributions based on [0, 1] Truncated Fréchet distribution. [0, 1] Truncated Fréchet Weibull ([0, 1]TFW) and [0, 1] Truncated Fréchet ([0, 1] TFF) distributions are discussed as special cases. The cumulative distribution function, the rth moment, the mean, the variance, the skewness, the kurtosis, the mode, the median, the characteristic function, the reliability function and the hazard rate function are obtained for the distributions under consideration. It is well known that an item fails when a stress to which it is subjected exceeds the corresponding strength. In this sense, strength can be viewed as "resistance to failure." Good design practice is such that the strength is always greater than the expected stress. The safety factor can be defined in terms of strength and stress as strength/stress. So, the [0, 1] TFW strength-stress and the [0, 1] TFF strength-stress models with different parameters will be derived here. The Shannon entropy and Relative entropy will be derived also.

Keywords: [0, 1] TFW, [0, 1] TFF, Stress-strength model, Shannon entropy, Relative entropy.

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1. Introduction

Probability distribution analysis of lifetime data is an important topic in reliability, biomedical, finance, social sciences and others. Lately, efforts have been made to derive new classes of statistical distributions that expand famed families of distributions and then get major pliancy in modeling data in practical applications. Therefore, by appending new parameters to a deriving distribution we obtain classes of more flexible distributions. Here, we propose a distribution with the hope it will attract wider applicability in other fields. The generalization, which is motivated by the work of Eugene et al., will be our guide. Eugene et al. [2] defined the beta G distribution from a quite arbitrary Cumulative Distribution Function (CDF), G(x) by

$$F(x) = (1/\beta(a,b)) \int_0^{G(x)} w^{a-1} (1-w)^{b-1} \, dw, \tag{1}$$

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where a > 0 and b > 0 are two additional parameters whose role are to introduce skewness and to vary tail weight, and $\beta(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}\,dw$ is the beta function. The class of distributions (1) has an increased attention after the works by Eugene et al. and Jones [2, 5]. Application of $X = G^{-1}(V)$ to the random variable V following a beta distribution with parameters a and b, V ~ B (a, b) say, and yields X with CDF (1). Eugene et al. [2] defined the Beta Normal (BN) distribution by taking G(x) to be the CDF of the normal distribution and derived some of its first moments. General expressions for the moments of the BN distribution were derived [4]. An extensive review of scientific literature on this subject is available in Abid and Hassan [1]. We can write (1) as

$$F(x) = I_{\mathrm{G}(x)}(a,b), \tag{2}$$

where $I_y(a,b) = (1/B(a,b)\int_0^y w^{a-1}(1-w)^{b-1}\,dw$ denotes the incomplete beta function ratio, i.e. the CDF of the beta distribution with parameters a and b. For general a and b, we can express (2) in terms of the well-known hypergeometric function defined by,

$${}_2F_1(\alpha,\beta,\gamma;x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i\,(\beta)_i}{(\gamma)_i\, i!}\, x^i,$$

where $(\alpha)_i = \alpha(\alpha+1)\ldots(\alpha+i-1)$ denotes the ascending factorial. We obtain

$$F(x) = \frac{\mathrm{G}(x)^a}{a\,B(a,b)}\ {}_2F_1(a, 1-b, a+1; \mathrm{G}(x)).$$

Based on the properties of the CDF, F(x) for any beta G distribution defined from a parent $\mathrm{G}(x)$ in (1), could, in principle, follow the properties of the hypergeometric function, which are well established in the literature (See Section 9.1 of Gradshteyn and Ryzhik [3]). The Probability Density Function (PDF) corresponding to (1) can be written in the form

$$f(x) = \frac{1}{B(a,b)}\mathrm{G}(x)^{a-1}\left(1-\mathrm{G}(x)\right)^{b-1} g(x), \tag{3}$$

where $g(x) = \partial\mathrm{G}(x)/\partial x$ is the PDF of the parent distribution.

Now, the PDF and CDF of [0, 1] truncated Fréchet distribution are respectively

$$h(x) = \frac{ab}{e^{-a}}\, x^{-(b+1)}\, e^{-ax^{-b}} \qquad 0 < x < 1, \tag{4}$$

$$H(x) = \frac{1}{e^{-a}}\, e^{-ax^{-b}}. \tag{5}$$

Graphs for some arbitrary parameters values of PDF and CDF are shown in Figure (1) and Figure (2), respectively

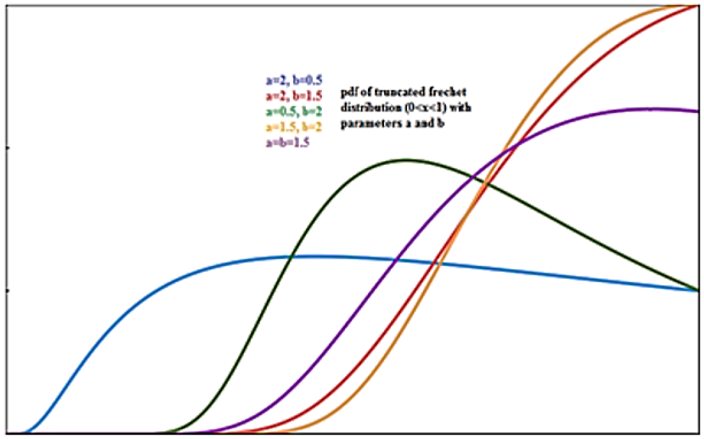

Figure 1. PDF of [0, 1] truncated Fréchet distribution.

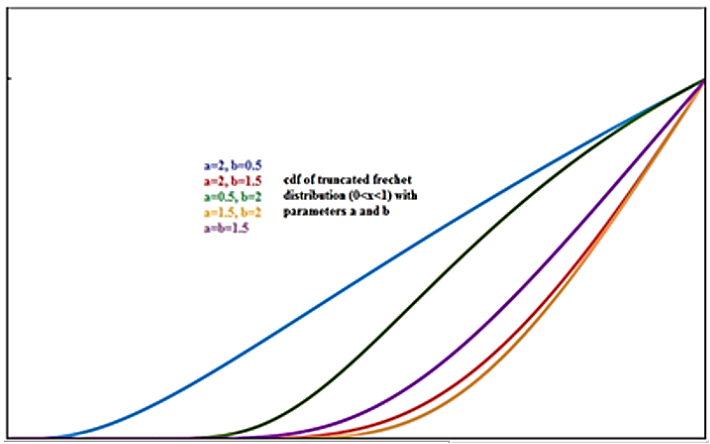

Figure 2. CDF of [0, 1] truncated Fréchet distribution.

Now, Given two absolutely continuous CDFs , H and G, so that H and G are their corresponding PDFs. We suggest a new distribution F by composing H with G, so that $F(x) = H(G(x))$ is a CDF:

$$F(x) = \int_0^{G(x)} \frac{ab}{e^{-a}} t^{-(b+1)} e^{-at^{-b}} dt \qquad (6)$$

$$= \frac{1}{e^{-a}} e^{-at^{-b}}\Big]_0^{G(x)} = \frac{1}{e^{-a}} e^{-aG(x)^{-b}},$$

with PDF:

$$f(x) = \frac{\partial}{\partial x} \mathrm{F(x)} = \frac{\partial}{\partial x}\frac{e^{-aG(x)^{-b}}}{e^{-a}} \qquad (7)$$

$$= \frac{ab}{e^{-a}} e^{-aG(x)^{-b}} \left(G(x)\right)^{-(b+1)} g(x).$$

By considering $G(x)$ as a baseline distribution, we define a generalized class of distributions in (6) and (7). We will name it the [0, 1] truncated Fréchet -G distribution. In the following two sections, we will assume that G are Weibull and Fréchet distributions, respectively.

2. [0, 1] Truncated Fréchet-Weibull Distribution

Assume that $g(x)=k/\lambda\ (x/\lambda)^{k-1}\,\mathrm{Exp}\{-(x/\lambda)^k\}$ and $G(x)=1-\mathrm{Exp}\{-(x/\lambda)^k\}$ $(0\le x)$ are PDF and CDF of Weibull random variable, respectively. Then, by applying (6) and (7), we get the CDF and PDF [0, 1] TFW random variable as follows:

$$F(x)=\frac{1}{e^{-a}}\,e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} \tag{8}$$

$$f(x)=\frac{abk}{\lambda e^{-a}}\left(\frac{x}{\lambda}\right)^{k-1}\,e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)}\,e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}\qquad x\ge 0\,. \tag{9}$$

So, the reliability $R(x)$ and hazard rate $\lambda(x)$ functions are respectively

$$R(x)=1-\frac{e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}}{e^{-a}}=1-e^{-a\left[\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}-1\right]},$$

$$\lambda(x)=\frac{\frac{abk}{\lambda e^{-a}}\left(\frac{x}{\lambda}\right)^{k-1}\,e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)}\,e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}}{1-e^{-a\left[\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}-1\right]}}\,.$$

The rth raw moment can be derived as follow:

$$E(x^r)=\int_0^\infty x^r\,\frac{abk}{\lambda e^{-a}}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)}e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}\,dx$$

$$=\frac{abk}{\lambda e^{-a}}\int_0^\infty \lambda^{-(k-1)}\,x^r\,x^{k-1}\,e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)}e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}\,dx.$$

By using poison series, $e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^m\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bm}$, we get

$$E(x^r)=\frac{abk}{\lambda e^{-a}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^m\int_0^\infty \lambda^{-k+1}\,x^{r+k-1}e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bm}dx$$

$$=\frac{bk}{\lambda^k e^{-a}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\,a^{m+1}\int_0^\infty x^{r+k-1}\,e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bm+b+1)}dx.$$

By using the series expansion, $(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{j!\,\Gamma(k)}z^j\quad |z|<1, k>0$, we get

$\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bm+b+1)}=\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}e^{-j\left(\frac{x}{\lambda}\right)^k}$, and then

$$E(x^r)=\frac{bk}{\lambda^k e^{-a}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\int_0^{\infty}x^{r+k-1}\,e^{-\left(\frac{x}{\lambda}\right)^k}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}e^{-j\left(\frac{x}{\lambda}\right)^k}dx$$

$$=\frac{bk}{\lambda^k e^{-a}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\int_0^{\infty}x^{r+k-1}\ e^{-\left(\frac{x}{\lambda}\right)^k}\ e^{-j\left(\frac{x}{\lambda}\right)^k}dx$$

$$E(x^r)=\frac{bk}{\lambda^k e^{-a}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\int_0^{\infty}x^{r+k-1}\ e^{-(j+1)\left(\frac{x}{\lambda}\right)^k}dx.$$

Let $y=\left(\frac{x}{\lambda}\right)^k \Rightarrow x=\lambda y^{\frac{1}{k}} \longrightarrow dx=\frac{\lambda}{k}\ y^{\frac{1}{k}-1}\,dy$, then

$$E(x^r)$$

$$=\frac{bk}{\lambda^k e^{-a}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\int_0^{\infty}\lambda^{r+k-1}\,y^{\frac{r}{k}+1-\frac{1}{k}}\,e^{-(j+1)y}\ \frac{\lambda}{k}\ y^{\frac{1}{k}-1}\,dy$$

$$=be^a\lambda^r\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\int_0^{\infty}y^{\frac{r}{k}}\ e^{-(j+1)y}dy$$

$$E(x^r)=be^a\lambda^r\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\ a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\ \frac{\Gamma\left(\frac{r}{k}+1\right)}{(j+1)^{\frac{r}{k}+1}}. \tag{10}$$

Then, the characteristic function is

$$Q_x(t)=E\left(e^{ixt}\right)$$

$$=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(x^r)\quad,\quad since\ \ e^{ixt}=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\ x^r$$

$$=be^a\lambda^r\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{r}{k}+1\right)}{(j+1)^{\frac{r}{k}+1}}$$

$$=be^a\sum_{r=0}^{\infty}\frac{(it\lambda)^r}{r!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{r}{k}+1\right)}{(j+1)^{\frac{r}{k}+1}}.$$

So, the mean μ and variance σ^2 of the of [0, 1] TFW random variable are

$$\mu=E(x)=be^a\lambda\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\ a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\ \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}. \tag{11}$$

$$\sigma^2=E(x^2)-(Ex)^2$$

$$=be^a\lambda^2\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\ a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\ \frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}}-b^2e^{2a}\lambda^2$$

$$\left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\ a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2.$$

$$\sigma^2 = be^a\lambda^2\left[\begin{array}{l}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}} - be^a\\ \left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\;a^{m+1}\;\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2\end{array}\right]. \quad (12)$$

Since $F(x) = \frac{e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}}{e^{-a}} = \frac{1}{2}$, the median M_e can be obtained as

$$x = M_e = \lambda\left\{-\ln\left[1-\left(1+\frac{\ln(2)}{a}\right)^{\frac{-1}{b}}\right]\right\}^{\frac{1}{k}}. \quad (13)$$

The skewness of [0, 1] TFW random variable will be $sk = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{Ex^3-3\mu Ex^2+2\mu^3}{(\sigma^2)^{3/2}}$

$$= \frac{\left(\begin{array}{l} be^a\lambda^3\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\;\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{3}{k}+1\right)}{(j+1)^{\frac{3}{k}+1}} - 3\\ \left\{be^a\lambda\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\;\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}\\ \left\{be^a\lambda^2\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\;\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}}\right\} +\\ 2\left\{be^a\lambda\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\;\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^3\end{array}\right)}{\left\{be^a\lambda^2\left[\begin{array}{l}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}} -\\ be^a\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\;\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right)^2\end{array}\right]\right\}^{3/2}} \quad (14)$$

In addition, the kurtosis is $kr = \frac{\mu_3}{{\mu_2}^2} - 3 = \frac{Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 3\mu^4}{(\sigma^2)^2} - 3$

$$
= \frac{\left\{\begin{array}{c}
be^a\lambda^4 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{4}{k}+1\right)}{(j+1)^{\frac{4}{k}+1}} \\
-4\left\{be^a\lambda \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\} \\
\left\{be^a\lambda^3 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\ \Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{3}{k}+1\right)}{(j+1)^{\frac{3}{k}+1}}\right\} \\
+6\left\{be^a\lambda \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2 \\
\left\{be^a\lambda^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}}\right\} \\
-3\left\{be^a\lambda \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^4
\end{array}\right\}}{\left\{\begin{array}{c}
be^a\lambda^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}} \\
-b^2e^{2a}\lambda^2\left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2
\end{array}\right\}^2} - 3
$$

$$= \frac{b^4e^{4a}\lambda^4\left\{\begin{array}{c} \frac{1}{b^3e^{3a}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{4}{k}+1\right)}{(j+1)^{\frac{4}{k}+1}} \\ -\frac{4}{b^2e^{2a}}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\} \\ \left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{3}{k}+1\right)}{(j+1)^{\frac{3}{k}+1}}\right\}+\frac{6}{be^a} \\ \left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2 \\ \left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}}\right\}-3 \\ \left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^4 \end{array}\right\}}{b^4e^{4a}\lambda^4\left\{\begin{array}{c} \frac{1}{be^a}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\;\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}} \\ -\left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2 \end{array}\right\}^2}-3$$

$$
= \frac{\left\{\begin{array}{c}
\frac{1}{b^3 e^{3a}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{4}{k}+1\right)}{(j+1)^{\frac{4}{k}+1}} - \\
\frac{4}{b^2 e^{2a}} \left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\} \\
\left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{3}{k}+1\right)}{(j+1)^{\frac{3}{k}+1}}\right\} + \frac{6}{be^a} \\
\left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2 \\
\left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}}\right\} - 3 \\
\left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^4
\end{array}\right\}}{\left\{\begin{array}{c}
\frac{1}{be^a} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{2}{k}+1\right)}{(j+1)^{\frac{2}{k}+1}} - \\
\left\{\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{1}{k}+1\right)}{(j+1)^{\frac{1}{k}+1}}\right\}^2
\end{array}\right\}^2} - 3 \qquad (15)
$$

The quintile function x_q of [0, 1] TFW random variable can be derived as

$$
q = P\left(x \le x_q\right) = F_x\left(x_q\right) = \frac{e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}}{e^{-a}} \qquad 0 < q < 1 \qquad x_q > 0
$$

$$
x_q = F^{-1}(q) = \lambda \left\{-\ln\left[1-\left(1-\frac{\ln(q)}{a}\right)^{\frac{-1}{b}}\right]\right\}^{\frac{1}{k}}. \qquad (16)
$$

So, by using the inverse transform method, we can generate [0, 1] TFW random variable as follows: $x = \lambda\left\{-\ln\left[1-\left(1-\frac{\ln(u)}{a}\right)^{\frac{-1}{b}}\right]\right\}^{\frac{1}{k}}$, where u is a random number distribution uniformly in the unit interval [0, 1].

2.1. Shannon and Relative Entropies

An entropy of a random variable X is a measure of variation of the uncertainty. The Shannon entropy of $[0,1]$ TFW(a,b,λ,k) random variable X can be found as follows:

$$H = -\int_0^\infty f(x)\ln\big(f(x)\big)\,dx$$

$$= -\int_0^\infty f(x)\ln\left(\frac{abk}{\lambda e^{-a}}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}\right)dx$$

$$= \ln\left(\frac{\lambda e^{-a}}{abk}\right) - (k-1)E\left(\ln\left(\frac{x}{\lambda}\right)\right) + E\left(\left(\frac{x}{\lambda}\right)^k\right) + (b+1)E\left(\ln\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)\right)$$

$$+ aE\left(\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}\right).$$

Let $I_1 = -(k-1)E\left(\ln\left(\frac{x}{\lambda}\right)\right)$,

$$H = -(k-1)\frac{abk}{\lambda e^{-a}}\int_0^\infty \ln\left(\frac{x}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}\,dx.$$

Since $e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} = \sum_{i=0}^\infty \frac{(-1)^i}{i!}a^i\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bi}$, then

$$I_1 = -(k-1)\frac{abk}{\lambda e^{-a}}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^i\int_0^\infty \ln\left(\frac{x}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bi}dx$$

$$I_1 = -(k-1)\frac{bk}{\lambda e^{-a}}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^{i+1}\int_0^\infty \ln\left(\frac{x}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)}dx.$$

By using expansion series $(1-z)^{-k} = \sum_{j=0}^\infty \frac{\Gamma(k+j)}{j!\Gamma(k)}z^j \quad |z|<1, k>0$, we get $\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-[b(i+1)+1]} = \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}e^{-j\left(\frac{x}{\lambda}\right)^k}$, and then

$$I_1 = -(k-1)\frac{bk}{\lambda e^{-a}}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\int_0^\infty \ln\left(\frac{x}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} e^{-j\left(\frac{x}{\lambda}\right)^k}dx$$

$$= -(k-1)\frac{bk}{\lambda e^{-a}}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\int_0^\infty \ln\left(\frac{x}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k}dx.$$

Let $y = \left(\frac{x}{\lambda}\right)^k \Longrightarrow x = \lambda y^{\frac{1}{k}} \Longrightarrow dx = \frac{\lambda}{k}y^{\frac{1}{k}-1}\,dy$, then

$$I_1 = -(k-1)\frac{bk}{\lambda e^{-a}}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\int_0^\infty \ln\left(y^{\frac{1}{k}}\right)\left(\frac{\lambda y^{\frac{1}{k}}}{\lambda}\right)^{k-1} e^{-(j+1)y}\,\frac{\lambda}{k}y^{\frac{1}{k}-1}dy$$

$$= \frac{-(k-1)be^a}{k}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\int_0^\infty \ln(y)\,e^{-(j+1)y}dy.$$

Since $\int_0^\infty x^{s-1}\, e^{-mx}\, ln(x)\, dx = m^{-s}\Gamma(s)\{\Psi(s) - ln(m)\}$, where
$\Psi(1) = -\ln(\gamma) \sim 0.5772$ and $\gamma = 0.5772$, then
$I_1 = \frac{(k-1)be^a}{k}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(j+1)}\{\gamma + \ln(j+1)\}$, and $I_2 = E\left(\left(\frac{x}{\lambda}\right)^k\right)$

$$= \frac{abk}{\lambda e^{-a}}\int_0^\infty \left(\frac{x}{\lambda}\right)^k \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx$$

$$= \frac{abk}{\lambda e^{-a}}\int_0^\infty \left(\frac{x}{\lambda}\right)^{2k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx.$$

Since $e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bi}$, then

$$I_2 = \frac{bk}{\lambda e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^\infty \left(\frac{x}{\lambda}\right)^{2k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} dx.$$

By using expansion series $(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{j!\Gamma(k)}z^j \quad |z| < 1\ , k > 0$, we get
$\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} = \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}e^{-j\left(\frac{x}{\lambda}\right)^k}$, and then

$$I_2 = \frac{bk}{\lambda e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\int_0^\infty \left(\frac{x}{\lambda}\right)^{2k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx.$$

Let $y = \left(\frac{x}{\lambda}\right)^k \longrightarrow x = \lambda y^{\frac{1}{k}} \longrightarrow dx = \frac{\lambda}{k}y^{\frac{1}{k}-1}dy$

$$I_2 = \frac{bk}{\lambda e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\int_0^\infty \left(\frac{\lambda y^{\frac{1}{k}}}{\lambda}\right)^{2k-1} e^{-(j+1)y}\frac{\lambda}{\kappa}\, y^{\frac{1}{k}-1}dy$$

$$= be^a\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\int_0^\infty y\ e^{-(j+1)y}dy$$

$$= be^a\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\frac{1}{(j+1)^2},$$

and $I_3 = (b+1)E\left(\ln\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)\right)$

$$= \frac{(b+1)abk}{\lambda e^{-a}}\int_0^\infty \ln\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx.$$

Let $y = \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b} \Rightarrow x = \lambda\left[-\ln\left(1 - y^{\frac{-1}{b}}\right)\right]^{\frac{1}{k}}$

$$\Longrightarrow dx = \frac{\lambda}{k}\left[-\ln\left(1-y^{\frac{-1}{b}}\right)\right]^{\frac{1}{k}-1}\frac{-1}{\left(1-y^{\frac{-1}{b}}\right)b}y^{\frac{-1}{b}-1}\,dy$$

$$\Longrightarrow dx = \frac{\lambda}{k}\left[-\ln\left(1-y^{\frac{-1}{b}}\right)\right]^{\frac{1}{k}-1}\frac{-1}{\left(1-y^{\frac{-1}{b}}\right)b}y^{\frac{-1}{b}-1}\,dy$$

$$I_3 = \frac{(b+1)abk}{\lambda e^{-a}}\int_1^{\infty}\ln\left(y^{\frac{-1}{b}}\right)\left(\lambda\left[-\ln\left(1-y^{\frac{-1}{b}}\right)\right]^{\frac{1}{k}}\Big/\lambda\right)^{k-1}\left(1-y^{\frac{-1}{b}}\right)\left(y^{\frac{-1}{b}}\right)^{-(b+1)}e^{-ay}$$

$$\frac{\lambda}{k}\left[-\ln\left(1-y^{\frac{-1}{b}}\right)\right]^{\frac{1}{k}-1}\frac{y^{\frac{-1}{b}-1}}{b\left(1-y^{\frac{-1}{b}}\right)}dy$$

$$= \frac{-(b+1)ae^{a}}{b}\int_1^{\infty}\ln(y)\ e^{-ay}\,dy$$

$$= \frac{-(b+1)ae^{a}}{b}\left[\int_0^{\infty}\ln(y)\,e^{-ay}dy - \int_0^{1}\ln(y)\,e^{-ay}dy\right]$$

$$I_{31} = \int_0^{\infty}\ln(y)\ e^{-ay}\,dy = \frac{-1}{a}\,\{\gamma+\ln(a)\}\,.$$

Since $\int_0^{\infty}x^{s-1}e^{-mx}\ln(x)\,dx = m^{-s}\Gamma(s)\{\Psi(s)-\ln(m)\}$, where $\Psi(1) = -\gamma$, $\gamma = 0.57721$ is an Eualer constant.

$I_{32} = \int_0^{1}\ln(y)\,e^{-ay}dy = \sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\int_0^1 y^m\ \ln(y)\ dy = -\sum_{m=0}^{\infty}\frac{(-a)^m}{m!(m+1)^2}$, since

$e^{-ay} = \sum_{m=0}^{\infty}\frac{(-a)^m}{m!}y^m$ and $\int x^m\ \ln(x)\ dx = x^{m+1}\left\{\frac{\ln(x)}{m+1}-\frac{1}{(m+1)^2}\right\}$,

$I_3 = \frac{(b+1)e^a}{b}\left\{\gamma+\ln(a)-\sum_{m=0}^{\infty}\frac{(-1)^m a^{m+1}}{m!(m+1)^2}\right\}$, and $I_4 = a\,\mathrm{E}\left(\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}\right)$

$$= \frac{a^2bk}{\lambda e^{-a}}\int_0^{\infty}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)}e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}\ \mathrm{dx}.$$

Since $e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bi}$, then

$$I_4 = \frac{bk}{\lambda e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\int_0^{\infty}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+2b+1)}\ \mathrm{dx}.$$

By using expansion series $(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{j!\Gamma(k)}z^j$ $|z|<1$ $,k>0$, we get

$\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+2b+1)} = \sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\ e^{-j\left(\frac{x}{\lambda}\right)^k}$, then

$$I_4 = \frac{bk}{\lambda e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}\int_0^{\infty}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(j+1)\left(\frac{x}{\lambda}\right)^k}\mathrm{dx}.$$

Let $y=\left(\frac{x}{\lambda}\right)^{k} \longrightarrow x=\lambda y^{\frac{1}{k}} \longrightarrow dx=\frac{\lambda}{k} y^{\frac{1}{k}-1} dy$

$$I_4=\frac{bk}{\lambda e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}\int_0^{\infty}\left(y^{\frac{1}{k}}\right)^{k-1}e^{-(j+1)y}\frac{\lambda}{k}y^{\frac{1}{k}-1}\,dy$$
$$=be^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}\int_0^{\infty}e^{-(j+1)y}\,dy$$
$$=be^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}\frac{1}{(j+1)}$$

$$H=\ln\left(\frac{\lambda e^{-a}}{abk}\right)+\frac{(k-1)}{k}be^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\frac{1}{(j+1)}\{\gamma+\ln(j+1)\}$$
$$+be^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\frac{1}{(j+1)^2}+\frac{(b+1)}{b}e^{a}$$
$$\left\{\gamma+\ln(a)-\sum_{m=0}^{\infty}\frac{(-1)^m\,a^{m+1}}{m!\,(m+1)^2}\right\}+b\,e^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}\frac{1}{(j+1)}. \quad (17)$$

The relative entropy or the Kullback–Leibler divergence is a measure of the difference between two probability, distributions F_1 and F_2. It is not symmetric in F_1 and F_2. In applications, F_1 typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while F_2 typically represents a theory, model, description, or approximation of F_1. Specifically, the Kullback–Leibler divergence of F_2 from F_1, denoted $D_{KL}(F_1\|F_2)$, is a measure of the information gained when one revises beliefs from the prior probability distribution F_2 to the posterior probability distribution F_1. More exactly, it is the amount of information that is *lost* when F_2 is used to approximate F_1, defined operationally as the expected extra number of bits required to code samples from F_1 using a code optimized for F_2 rather than the code optimized for F_1. So, the relative entropy $Dkl(F\|F^*)$ for a random variable [0, 1] TFW(a,b,λ,k) can be found as follows:

$$\frac{f(x)}{f^*(x)}=\frac{abk\lambda_1 e^{-\alpha}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-(b+1)}e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-b}}}{\alpha\beta k_1\lambda e^{-a}\left(\frac{x}{\lambda_1}\right)^{k_1-1}e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-(\beta+1)}e^{-\alpha\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}}}$$

$$Dkl(F\|F^*)$$

$$= \int_0^\infty f(x) \ln \left(\frac{abk\lambda_1 e^{-\alpha} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}}}{\alpha\beta k_1 \lambda e^{-a} \left(\frac{x}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-(\beta+1)} e^{-\alpha\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}}} \right) dx$$

$$= \int_0^\infty f_1(x) \left[\begin{array}{c} \ln\left(\frac{abk\lambda_1 e^{-\alpha}}{\alpha\beta k_1 \lambda e^{-a}}\right) + (k-1)\ln\left(\frac{x}{\lambda}\right) - \left(\frac{x}{\lambda}\right)^k - (b+1)\ln\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right) - \\ a\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b} - (k_1-1)\ln\left(\frac{x}{\lambda_1}\right) + \left(\frac{x}{\lambda_1}\right)^{k_1} + (\beta+1)\ln\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right) \\ +\alpha\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta} \end{array} \right] dx$$

$$= \ln\left(\frac{abk\lambda_1 e^{-\alpha}}{\alpha\beta k_1 \lambda e^{-a}}\right) + (k-1)E\left(\ln\left(\frac{x}{\lambda}\right)\right) - E\left(\left(\frac{x}{\lambda}\right)^k\right) - (b+1)E\left(\ln\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)\right) - a$$
$$E\left(\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}\right) - (k_1-1)E\left(\ln\left(\frac{x}{\lambda_1}\right)\right) + E\left(\left(\frac{x}{\lambda_1}\right)^{k_1}\right) + (\beta+1)E\left(\ln\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)\right) +$$
$$\alpha E\left(\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}\right).$$

Let, $I_1 = (k-1)E\left(\ln\left(\frac{x}{\lambda}\right)\right)$

$$= (k-1)\frac{abk}{\lambda e^{-a}} \int_0^\infty \ln\left(\frac{x}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx$$

$$= -\frac{(k-1)}{k} b e^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(j+1)} \{\gamma + \ln(j+1)\},$$

and $I_2 = -E\left(\left(\frac{x}{\lambda}\right)^k\right)$

$$= \frac{-abk}{\lambda e^{-a}} \int_0^\infty \left(\frac{x}{\lambda}\right)^k \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx$$

$$= -b e^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(j+1)^2},$$

and $I_3 = -(b+1)E\left(\ln\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)\right)$

$$= -(b+1)\int_0^\infty \ln\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right) \frac{abk}{\lambda e^{-a}} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx$$

$$= \frac{(b+1)}{b} e^{a} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m \, a^{m+1}}{m! \, (m+1)^2} - (\gamma + \ln(a)) \right\},$$

and $I_4 = -a \quad E\left(\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}\right)$

$$= -a \int_0^{\infty} \left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b} \frac{abk}{\lambda e^{-a}} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx$$

$$= -b\, e^{a} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+2} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])} \frac{1}{(j+1)},$$

and $I_5 = -(k_1 - 1) E\left(\ln\left(\frac{x}{\lambda_1}\right)\right)$

$$= -(k_1-1) \frac{abk}{\lambda e^{-a}} \int_0^{\infty} \ln\left(\frac{x}{\lambda_1}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx.$$

Since $e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bi}$, then

$$I_5 = -(k_1-1) \frac{bk}{\lambda e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} \ln\left(\frac{x}{\lambda_1}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} dx.$$

By using expansion series $(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\Gamma(k)} z^j \quad |z| < 1, k > 0$, we get

$\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} = \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} e^{-j\left(\frac{x}{\lambda}\right)^k}$, and then

$$I_5 = -(k_1-1) \frac{bk}{\lambda e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \int_0^{\infty} \ln\left(\frac{x}{\lambda_1}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx.$$

Let $y = \left(\frac{x}{\lambda}\right)^k \longrightarrow x = \lambda\, y^{\frac{1}{k}} \longrightarrow dx = \frac{\lambda}{k} y^{\frac{1}{k}-1} dy$

$$I_5 = -(k_1 - 1) \frac{bk}{\lambda e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \int_0^{\infty} \ln\left(\frac{\lambda y^{\frac{1}{k}}}{\lambda_1}\right) \left(y^{\frac{1}{k}}\right)^{k-1} e^{-(j+1)y} \frac{\lambda}{k} y^{\frac{1}{k}-1} dy$$

$$= -(k_1-1) b e^{a} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \int_0^{\infty} \left[\ln\left(\frac{\lambda}{\lambda_1}\right) + \frac{1}{k} \ln y\right] e^{-(j+1)y} dy$$

$$= (k_1-1)\, b\, e^{a} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \left\{ \frac{1}{k(j+1)} (\gamma + \ln(j+1)) - \ln\left(\frac{\lambda}{\lambda_1}\right) \frac{1}{(j+1)} \right\},$$

and $I_6 = E\left(\left(\frac{x}{\lambda_1}\right)^{k_1}\right) = \left(\frac{1}{\lambda_1}\right)^{k_1} E\left(x^{k_1}\right)$

$$= \left(\frac{1}{\lambda_1}\right)^{k_1} be^a \lambda^{k_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{k_1}{k}+1\right)}{(j+1)^{\frac{k_1}{k}+1}}$$

$$= be^a \left(\frac{\lambda}{\lambda_1}\right)^{k_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])} \frac{\Gamma\left(\frac{k_1}{k}+1\right)}{(j+1)^{\frac{k_1}{k}+1}},$$

and $I_7 = (\beta+1) E\left(\ln\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)\right)$

$$= (\beta+1) \frac{abk}{\lambda e^{-a}} \int_0^{\infty} \ln\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx.$$

Since $e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-bi}$, then

$$I_7 = (\beta+1) \frac{bk}{\lambda e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} \ln\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} dx$$

$\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} = \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} e^{-j\left(\frac{x}{\lambda}\right)^k}$, and then

$$I_7 = (\beta+1) \frac{bk}{\lambda e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \int_0^{\infty} \left(\frac{x}{\lambda}\right)^{k-1} \ln\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right) e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx$$

$$\Rightarrow let\ y = 1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} \Rightarrow x = \lambda_1[-\ln(1-y)]^{\frac{1}{k_1}} \Rightarrow dx = \frac{\lambda_1}{k_1}[-\ln(1-y)]^{\frac{1}{k_1}-1} \frac{1}{(1-y)} dy,$$

then

$$I_7 = \frac{(\beta+1)bk}{\lambda e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \int_0^1 \ln(y) \left(\frac{\lambda_1[-\ln(1-y)]^{\frac{1}{k_1}}}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{\lambda_1[-\ln(1-y)]^{\frac{1}{k_1}}}{\lambda}\right)^k} \frac{\lambda_1}{k_1}[-\ln(1-y)]^{\frac{1}{k_1}-1} \frac{1}{1-y} dy$$

$$= \frac{(\beta+1)bk}{\lambda e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \left(\frac{\lambda_1}{\lambda}\right)^{k-1} \left(\frac{\lambda_1}{k_1}\right) \int_0^1 \ln(y)\ [-\ln(1-y)]^{\frac{k}{k_1}-1} e^{-(j+1)\left(\frac{\lambda_1}{\lambda}\right)^k [-\ln(1-y)]^{\frac{k}{k_1}}} \frac{1}{(1-y)} dy.$$

Since $e^{-(j+1)\left(\frac{\lambda_1}{\lambda}\right)^k[-\ln(1-y)]^{\frac{k}{k_1}}} = \sum_{v=0}^{\infty}\frac{(-1)^v}{v!}\left[(j+1)\left(\frac{\lambda_1}{\lambda}\right)^k\right]^v[-\ln(1-y)]^{\frac{vk}{k_1}}$, then

$$I_7 = (\beta+1)\frac{bk}{\lambda e^{-a}}\sum_{i,v=0}^{\infty}\frac{(-1)^{i+v}}{i!\,v!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\left(\frac{\lambda_1}{\lambda}\right)^k\left(\frac{\lambda}{\lambda_1}\right)\left(\frac{\lambda_1}{k_1}\right)\left[(j+1)\left(\frac{\lambda_1}{\lambda}\right)^k\right]^v$$
$$\int_o^1 \ln(y)\ [-\ln(1-y)]^{\frac{k}{k_1}-1}[-\ln(1-y)]^{\frac{vk}{k_1}}(1-y)^{-1}\,dy$$
$$= (\beta+1)\frac{bk}{k_1 e^{-a}}\sum_{i,v=0}^{\infty}\frac{(-1)^{i+v}}{i!\,v!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\left(\frac{\lambda_1}{\lambda}\right)^k\left[(j+1)\left(\frac{\lambda_1}{\lambda}\right)^k\right]^v\int_0^1 \ln(y)$$
$$[-\ln(1-y)]^{\frac{vk}{k_1}+\frac{k}{k_1}-1}\,(1-y)^{-1}dy.$$

Since the expansion of $[-\ln(1-y)]^{\frac{vk}{k_1}+\frac{k}{k_1}-1} =$

$$\left(\frac{vk}{k_1}+\frac{k}{k_1}-1\right)\sum_{m=0}^{\infty}\binom{m+1-\frac{vk}{k_1}-\frac{k}{k_1}}{m}\sum_{u=0}^{m}\frac{(-1)^{m+u}}{\frac{vk}{k_1}+\frac{k}{k_1}-1-u}\binom{m}{u}p_{u,m}\,y^{\frac{vk}{k_1}+\frac{k}{k_1}+m-1},$$

where $p_{u,m}$ is constant can be calculated as

$$p_{u,m} = \frac{1}{m}\sum_{n=1}^{m}\frac{(-1)^n[n(u+1)-m]}{(n+1)}p_{u,m-n},\quad for\ m=1,2,3,\dots\quad and\ p_{u,0}=1.$$

By using $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$, we get $(1-y)^{-1} = 1 + y + y^2 + y^3 + \cdots = \sum_{r=0}^{\infty}y^r$, and then

$$I_7 = \frac{(\beta+1)bk}{k_1 e^{-a}}\sum_{i,v,r=0}^{\infty}\frac{(-1)^{i+v}}{i!\,v!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\left(\frac{\lambda_1}{\lambda}\right)^k\left[(j+1)\left(\frac{\lambda_1}{\lambda}\right)^k\right]^v$$
$$\left(\frac{vk}{k_1}+\frac{k}{k_1}-1\right)\sum_{m=0}^{\infty}\binom{m+1-\frac{vk}{k_1}-\frac{k}{k_1}}{m}\sum_{u=0}^{m}\frac{(-1)^{m+u}}{\frac{vk}{k_1}+\frac{k}{k_1}-1-u}\binom{m}{u}p_{u,m}\int_0^1\ln(y)\ y^{\frac{vk}{k_1}+\frac{k}{k_1}+m+r-1}dy.$$

Since $\int_0^1\left(\ln\left(\frac{1}{y}\right)\right)^{\theta-1}y^{v-1}dy = \frac{1}{v^\theta}\ \Gamma(\theta)$, then

$\int_0^1\ln(y)\ y^{\frac{vk}{k_1}+\frac{k}{k_1}+m+r-1}\,dy = -\int_0^1\ln\left(\frac{1}{y}\right)\ y^{\frac{vk}{k_1}+\frac{k}{k_1}+m+r-1}\,dy = \frac{1}{\left(\frac{vk}{v_1}+\frac{k}{k_1}+m+r\right)^2}\Gamma(2)$, so

$$I_7 = \frac{(\beta+1)bk}{k_1 e^{-a}}\sum_{i,v,r=0}^{\infty}\frac{(-1)^{i+v+1}}{i!\,v!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\left(\frac{\lambda_1}{\lambda}\right)^k\left[(j+1)\left(\frac{\lambda_1}{\lambda}\right)^k\right]^v$$
$$\left(\frac{vk}{k_1}+\frac{k}{k_1}-1\right)\sum_{m=0}^{\infty}\binom{m+1-\frac{vk}{k_1}-\frac{k}{k_1}}{m}\sum_{u=0}^{m}\frac{(-1)^{m+u}}{\frac{vk}{k_1}+\frac{k}{k_1}-1-u}\binom{m}{u}p_{u,m}\ \frac{1}{\left(\frac{vk}{k_1}+\frac{k}{k_1}+m+r\right)^2},$$

and $I_8 = \alpha E\left(\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}\right)$

$$I_8 = \frac{\alpha abk}{\lambda e^{-a}} \int_0^\infty \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(b+1)} e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-b}} dx$$

$$= \frac{\alpha bk}{\lambda e^{-a}} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \int_0^\infty \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} dx.$$

Since $\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} = \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} e^{-j\left(\frac{x}{\lambda}\right)^k}$, then

$$I_8 = \frac{\alpha bk}{\lambda e^{-a}} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \int_0^\infty \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx.$$

By using $\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta} = \sum_{m=0}^\infty \frac{\Gamma(\beta+m)}{m!\Gamma(\beta)} e^{-m\left(\frac{x}{\lambda_1}\right)^{k_1}}$, we get

$$I_8 = \frac{\alpha bk}{\lambda e^{-a}} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^\infty \frac{\Gamma(\beta+m)}{m!\Gamma(\beta)} \int_0^\infty e^{-m\left(\frac{x}{\lambda_1}\right)^{k_1}} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx.$$

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So, by using $e^{-m\left(\frac{x}{\lambda_1}\right)^{k_1}} = \sum_{l=0}^\infty \frac{(-1)^l}{l!} m^l \left(\frac{x}{\lambda_1}\right)^{lk_1}$, we get

$$I_8 = \frac{\alpha bk}{\lambda e^{-a}} \sum_{i=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+l}}{i!\, l!} a^{i+1} m^l \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \sum_{m=0}^\infty \frac{\Gamma(\beta+m)}{m!\,\Gamma(\beta)} \int_0^\infty \left(\frac{x}{\lambda_1}\right)^{lk_1} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx$$

$$= \frac{\alpha bke^a}{\lambda_1^{lk_1}\lambda^k} \sum_{i=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+l}}{i!\, l!} a^{i+1} m^l \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \sum_{m=0}^\infty \frac{\Gamma(\beta+m)}{m!\,\Gamma(\beta)} \int_0^\infty x^{lk_1+k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx$$

$$\Rightarrow let\ y = \left(\frac{x}{\lambda}\right)^k \Rightarrow x = \lambda y^{\frac{1}{k}} \Rightarrow dx = \frac{\lambda}{k} y^{\frac{1}{k}-1} dy, then$$

$$I_8 = \frac{\alpha be^a \lambda^{lk_1}}{\lambda_1^{lk_1}} \sum_{i=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+l}}{i!\, l!} a^{i+1} m^l \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \sum_{m=0}^\infty \frac{\Gamma(\beta+m)}{m!\,\Gamma(\beta)} \int_0^\infty y^{\frac{lk_1}{k}} e^{-(j+1)y} dy$$

$$= \alpha be^a \left(\lambda/\lambda_1\right)^{lk_1} \sum_{i=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+l}}{i!\, l!} a^{i+1} m^l \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \sum_{m=0}^\infty \frac{\Gamma(\beta+m)}{m!\,\Gamma(\beta)} \frac{\Gamma\left(\frac{lk_1}{k}+1\right)}{(j+1)^{\frac{lk_1}{k}+1}}.$$

Then, the relative entropy is

$$Dkl(F_1\|F_2) = \ln\left(\frac{abk\lambda_1 e^{-\alpha}}{\alpha\beta k_1\lambda e^{-a}}\right) - \frac{(k-1)}{k} be^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}$$

$$\frac{1}{(j+1)}\{\gamma+\ln(j+1)\}-be^{a}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\frac{1}{(j+1)^{2}}$$
$$+\frac{(b+1)}{b}e^{a}\left\{\sum_{m=0}^{\infty}\frac{(-1)^{m}a^{m+1}}{m!\,(m+1)^{2}}-(\gamma+\ln(a))\right\}$$
$$-\,be^{a}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b[i+2]+1])}$$
$$\frac{1}{(j+1)}+(k_1-1)be^{a}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+)+1])}\left\{\begin{matrix}\frac{1}{k(j+1)}(\gamma+\ln(j+1))\\ -\ln\left(\frac{\lambda}{\lambda_1}\right)\frac{1}{(j+1)}\end{matrix}\right\}$$
$$+be^{a}\left(\frac{\lambda}{\lambda_1}\right)^{k_1}\sum_{m=0}^{\infty}\frac{(-1)^{m}}{m!}a^{m+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(m+1)+1]+j)}{j!\,\Gamma([b(m+1)+1])}\frac{\Gamma\left(\frac{k_1}{k}+1\right)}{(j+1)^{\frac{k_1}{k}+1}}+\ (\beta+1)\frac{bk}{k_1e^{-a}}$$
$$\sum_{i,v,r=0}^{\infty}\frac{(-1)^{i+v+1}}{i!\,v!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\left(\frac{\lambda_1}{\lambda}\right)^{k}\left[(j+1)\left(\frac{\lambda_1}{\lambda}\right)^{k}\right]^{v}$$
$$\left(\frac{vk}{k_1}+\frac{k}{k_1}-1\right)\sum_{m=0}^{\infty}\binom{m+1-\frac{vk}{k_1}-\frac{k}{k_1}}{m}\sum_{u=0}^{m}\frac{(-1)^{m+u}}{\frac{vk}{k_1}+\frac{k}{k_1}-1-u}\binom{m}{u}p_{u,m}\frac{1}{\left(\frac{vk}{k_1}+\frac{k}{k_1}+m+r\right)^{2}}$$
$$+\alpha be^{a}\left(\frac{\lambda}{\lambda_1}\right)^{lk_1}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+l}}{i!\,l!}a^{i+1}m^{l}\sum_{j=0}^{\infty}\frac{\Gamma([b(j+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty}\frac{\Gamma(\beta+m)}{m!\,\Gamma(\beta)}\frac{\Gamma\left(\frac{lk_1}{k}+1\right)}{(j+1)^{\frac{lk_1}{k}+1}} \tag{18}$$

2.2. Stress –Strength Reliability

Let Y and X be the stress and strength random variables, independent of each other, follow [0,1] TFW(a,b,λ,k) and [0,1]TFW$(\alpha,\beta,\lambda_1,k_1)$, respectively, then

$$R=P(y<x)=\int_0^{\infty}f_X(x)\ F_Y(x)\,dx$$

$$=\int_0^{\infty}\frac{abk}{\lambda e^{-a}}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-(b+1)}e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-b}}\ \frac{e^{-\alpha\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}}}{e^{-\alpha}}\,dx.$$

Since $e^{-a\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-bi}$, then

$$R=\frac{abk}{\lambda e^{-a}e^{-\alpha}}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i}\int_0^{\infty}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-(b+1)}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-bi}e^{-\alpha\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}}dx$$

$$= \frac{bk}{\lambda e^{-a} e^{-\alpha}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} e^{-\alpha\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}} dx.$$

By using $\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-(bi+b+1)} = \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} e^{-j\left(\frac{x}{\lambda}\right)^k}$, we get

$$R = \frac{bk}{\lambda e^{-a} e^{-\alpha}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \int_0^{\infty} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} e^{-\alpha\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}} dx.$$

So, by using $e^{-\alpha\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \alpha^m \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta m}$, we get

$$R = \frac{bk}{\lambda e^{-a} e^{-\alpha}} \sum_{i=0}^{\infty} \frac{(-1)^{i+m}}{i!\,m!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)]+1)} \sum_{m=0}^{\infty} \alpha^m \int_0^{\infty} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta m} dx$$

$$= \frac{bk}{\lambda e^{-a} e^{-\alpha}} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m}}{i!\,m!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \alpha^m \sum_{u=0}^{\infty} \frac{\Gamma(\beta m+u)}{u!\,\Gamma(\beta m)} \int_0^{\infty} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} e^{-u\left(\frac{x}{\lambda_1}\right)^{k_1}} dx.$$

Since $e^{-u\left(\frac{x}{\lambda_1}\right)^{k_1}} = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} u^v \left(\frac{x}{\lambda_1}\right)^{vk_1}$, then

$$R = \frac{bk}{\lambda e^{-a} e^{-\alpha}} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \sum_{v=0}^{\infty} \frac{(-1)^{i+m+v}}{i!\,m!\,v!} a^{i+1} \alpha^m u^v \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \sum_{u=0}^{\infty} \frac{\Gamma(\beta m+u)}{u!\,\Gamma(\beta m)} \int_0^{\infty} \lambda_1^{-vk_1} \left(\frac{x}{\lambda}\right)^{k-1} x^{vk_1} e^{-(j+1)\left(\frac{x}{\lambda}\right)^k} dx.$$

Let $y = \left(\frac{x}{\lambda}\right)^k \longrightarrow x = \lambda y^{\frac{1}{k}} \longrightarrow dx = \frac{\lambda}{k} y^{\frac{1}{k}-1} dy$, then

$$R = \frac{bk}{\lambda e^{-a} e^{-\alpha}} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \sum_{v=0}^{\infty} \frac{(-1)^{i+m+v}}{i!\,m!\,v!} a^{i+1} \alpha^m u^v \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \sum_{u=0}^{\infty} \frac{\Gamma(\beta m+u)}{u!\,\Gamma(\beta m)} \int_0^{\infty} \lambda_1^{-vk_1} \left(y^{\frac{1}{k}}\right)^{k-1} \lambda^{vk_1} y^{\frac{vk_1}{k}} e^{-(j+1)y} \frac{\lambda}{k} y^{\frac{1}{k}-1} dy$$

$$= be^{a+\alpha} \left(\frac{\lambda}{\lambda_1}\right)^{vk_1} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \sum_{v=0}^{\infty} \frac{(-1)^{i+m+v}}{i!\,m!\,v!} a^{i+1} \alpha^m u^v \sum_{u=0}^{\infty} \frac{\Gamma(\beta m+u)}{u!\,\Gamma(\beta u)} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma\left(\frac{vk_1}{k}+1\right)}{(j+1)^{\frac{vk_1}{k}+1}}, \tag{19}$$

where $\left(1-e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^{-\beta m} = \sum_{u=0}^{\infty} \frac{\Gamma(\beta m+u)}{u!\Gamma(\beta m)}\, e^{-u\left(\frac{x}{\lambda_1}\right)^{k_1}}$.

3. [0, 1] Truncated Fréchet- Fréchet Distribution

We assume that $g(x) = \alpha\beta x^{-(\beta+1)}\mathrm{Exp}\{-\alpha x^{-\beta}\}$ and $G(x) = \mathrm{Exp}\{-\alpha x^{-\beta}\}$ $(0 < x)$ are PDF and CDF of Fr$\acute{e}$chet random variable, respectively. Then, by applying the equations (6) and (7), we get the PDF and CDF of [0, 1] TFF random variable as follows:

$$F(x) = \frac{e^{-ae^{\alpha b x^{-\beta}}}}{e^{-a}} \tag{20}$$

$$f(x) = \frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} \left(e^{-\alpha x^{-\beta}}\right)^{-b} e^{-a\left(e^{-\alpha x^{-\beta}}\right)^{-b}} \qquad x \geq 0$$

$$f(x) = \frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-ae^{\alpha b x^{-\beta}}} \qquad x \geq 0. \tag{21}$$

So, the reliability $R(x)$ and hazard rate $\lambda(x)$ functions are respectively

$$R(x) = 1 - \frac{e^{-ae^{\alpha b x^{-\beta}}}}{e^{-a}}$$
$$= 1 - e^{-a\left(e^{\alpha b x^{-\beta}}-1\right)}$$
$$\lambda(x) = \frac{f_1(x)}{R(x)} = \frac{ab\alpha\beta e^{a} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-ae^{\alpha b x^{-\beta}}}}{1 - e^{-a\left(e^{\alpha b x^{-\beta}}-1\right)}}.$$

The rth raw moment can be derived as follows:

$$E(x^r) = \int_0^{\infty} x^r \frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-ae^{\alpha b x^{-\beta}}} dx.$$

Let $y = e^{\alpha b x^{-\beta}} \Longrightarrow x = \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}} \Longrightarrow dx = \frac{-1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b}\frac{1}{y} dy$, then

$$E(x^r) = \frac{ab\alpha\beta}{e^{-a}} \int_1^{\infty} \left(\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}}\right)^{r-\beta-1} y e^{-ay} \frac{1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b y} dy$$
$$= ae^{a} \int_1^{\infty} \left(\frac{\ln y}{\alpha b}\right)^{\frac{-r}{\beta}+1+\frac{1}{\beta}} e^{-ay} \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} dy$$
$$= ae^{a} (\alpha b)^{\frac{r}{\beta}} \int_1^{\infty} (\ln y)^{\frac{-r}{\beta}} e^{-ay} dy$$
$$E(x^r) = ae^{a} (\alpha b)^{\frac{r}{\beta}} \tau_u\left(a, \frac{r}{\beta}, 1\right), \tag{22}$$

where $\int_0^{\infty} (\ln y)^{\frac{-r}{\beta}} e^{-ay} dy = \tau\left(a, \frac{r}{\beta}\right)$ *and* $\int_0^{1} (\ln y)^{\frac{-r}{\beta}} e^{-ay} dy = \tau_l\left(a, \frac{r}{\beta}, 1\right)$

$$E(x^r) = ae^a(\alpha b)^{\frac{r}{\beta}}\left[\int_0^\infty (\ln y)^{\frac{-r}{\beta}} e^{-ay}dy - \int_0^1 (\ln y)^{\frac{-r}{\beta}} e^{-ay}dy\right]$$

$$E(x^r) = ae^a(\alpha b)^{\frac{r}{\beta}}\left\{\tau\left(a,\frac{r}{\beta}\right) - \tau_l\left(a,\frac{r}{\beta},1\right)\right\},$$

and the characteristic function is

$$Q_x(t) = E\left(e^{itx}\right)$$

$$= \sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(x^r)\text{ , since } e^{ixt} = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}x^r$$

$$Q_x(t) = ae^a\sum_{r=0}^{\infty}\frac{\left(it(\alpha b)^{1/\beta}\right)^r}{r!}\ \tau_u\left(a,\frac{r}{\beta},1\right).$$

So, the mean μ and variance σ^2 of the of [0, 1] TFF random variable are

$$\mu = E(x) = ae^a(\alpha b)^{\frac{1}{\beta}}\,\tau_u\left(a,\frac{1}{\beta},1\right) \tag{23}$$

$$\sigma^2 = E(x^2) - (Ex)^2$$

$$\sigma^2 = a^2e^{2a}\,(\alpha b)^{\frac{2}{\beta}}\left\{\frac{1}{ae^a}\tau_u\left(a,\frac{2}{\beta},1\right) - \tau_u^2\left(a,\frac{1}{\beta},1\right)\right\}. \tag{24}$$

Since $F(x) = \frac{e^{-ae^{\alpha b x^{-\beta}}}}{e^{-a}} = \frac{1}{2}$, then the median M_e can be calculated as

$$x = M_e = \left(\frac{\ln(1+\ln(2)/a)}{\alpha b}\right)^{\frac{-1}{\beta}} \tag{25}$$

$$sk = \frac{\mu_3}{\mu_2^{3/2}} = \frac{Ex^3 - 3\mu Ex^2 + 2\mu^3}{(\sigma^2)^{3/2}}$$

$$sk = \frac{\left\{\begin{array}{c} ae^a\,(\alpha b)^{\frac{3}{\beta}}\,\tau_u\left(a,\frac{3}{\beta},1\right) - 3\left\{ae^a(\alpha b)^{\frac{1}{\beta}}\,\tau_u\left(a,\frac{1}{\beta},1\right)\right\} \\ \left\{ae^a(\alpha b)^{\frac{2}{\beta}}\,\tau_u\left(a,\frac{2}{\beta},1\right)\right\} + 2\left\{ae^a(\alpha b)^{\frac{1}{\beta}}\,\tau_u\left(a,\frac{1}{\beta},1\right)\right\}^3 \end{array}\right\}}{\left\{a^2e^{2a}(\alpha b)^{\frac{2}{\beta}}\left[\frac{1}{ae^a}\tau_u\left(a,\frac{2}{\beta},1\right) - \tau_u^2\left(a,\frac{1}{\beta},1\right)\right]\right\}^{3/2}} \tag{26}$$

$$kr = \frac{\mu_4}{\mu_2^2} - 3 = \frac{Ex^4 - 4\mu Ex^3 + 6\mu^2Ex^2 - 3\mu^4}{(\sigma^2)^2} - 3$$

$$kr = \frac{\left\{\begin{array}{c} ae^a(\alpha b)^{\frac{4}{\beta}}\,\tau_u\left(a,\frac{4}{\beta},1\right) - 4\left\{ae^a(\alpha b)^{\frac{1}{\beta}}\,\tau_u\left(a,\frac{1}{\beta},1\right)\right\} \\ \left\{ae^a(\alpha b)^{\frac{3}{\beta}}\,\tau_u\left(a,\frac{3}{\beta},1\right)\right\} + 6\left\{ae^a(\alpha b)^{\frac{1}{\beta}}\tau_u\left(a,\frac{1}{\beta},1\right)\right\}^2 \\ \left\{ae^a(\alpha b)^{\frac{2}{\beta}}\tau_u\left(a,\frac{2}{\beta},1\right)\right\} - 3\left\{ae^a(\alpha b)^{\frac{1}{\beta}}\tau_u\left(a,\frac{1}{\beta},1\right)\right\}^4 \end{array}\right\}}{\left\{a^2e^{2a}(\alpha b)^{\frac{2}{\beta}}\left[\frac{1}{ae^a}\tau_u\left(a,\frac{2}{\beta},1\right) - \tau_u^2\left(a,\frac{1}{\beta},1\right)\right]\right\}^2} - 3$$

$$= \frac{a^4 e^{4a} (\alpha b)^{\frac{4}{\beta}} \left\{ \begin{array}{c} \frac{1}{a^3 e^{3a}} \tau_u\left(a, \frac{4}{\beta}, 1\right) - \frac{4}{a^2 e^{2a}} \left\{\tau_u\left(a, \frac{1}{\beta}, 1\right)\right\} \\ \left\{\tau_u\left(a, \frac{3}{\beta}, 1\right)\right\} + \frac{6}{a e^a} \tau_u^2\left(a, \frac{1}{\beta}, 1\right)\left\{\tau_u\left(a, \frac{2}{\beta}, 1\right)\right\} \\ -3\tau_u^4\left(a, \frac{1}{\beta}, 1\right) \end{array} \right\}}{a^4 e^{4a} (\alpha b)^{\frac{4}{\beta}} \left\{\frac{1}{a e^a} \tau_u\left(a, \frac{2}{\beta}, 1\right) - \tau_u^2\left(a, \frac{1}{\beta}, 1\right)\right\}^2} - 3$$

$$kr = \frac{\left\{ \begin{array}{c} \frac{1}{a^3 e^{3a}} \tau_u\left(a, \frac{4}{\beta}, 1\right) - \frac{4}{a^2 e^{2a}} \tau_u\left(a, \frac{1}{\beta}, 1\right)\left\{\tau_u\left(a, \frac{3}{\beta}, 1\right)\right\} \\ + \frac{6}{a e^a} \tau_u^2\left(a, \frac{1}{\beta}, 1\right)\left\{\tau_u\left(a, \frac{2}{\beta}, 1\right)\right\} - 3\tau_u^4\left(a, \frac{1}{\beta}, 1\right) \end{array} \right\}}{\left\{\frac{1}{a e^a} \tau_u\left(a, \frac{2}{\beta}, 1\right) - \tau_u^2\left(a, \frac{1}{\beta}, 1\right)\right\}^2} - 3\,. \quad (27)$$

Since $\left(x \le x_q\right) = F_x\left(x_q\right) = \frac{e^{-ae^{\alpha b x^{-\beta}}}}{e^{-a}}$ $\quad 0 < q < 1 \quad x_q > 0$, then the quintile function x_q of [0, 1] TFF random variable can be defined as

$$x_q = F^{-1}(q) = \left[\frac{\ln\left(1 - \frac{\ln(q)}{a}\right)}{\alpha b}\right]^{\frac{-1}{\beta}}.$$

So, by using the inverse transform method, we can generate [0, 1] TFF random variable as follows:

$$x = \left[\frac{\ln\left(1 - \frac{\ln(u)}{a}\right)}{\alpha b}\right]^{\frac{-1}{\beta}},$$

where u is a uniformly distributed random number lie in the unit interval [0,1].

3.1. Shannon and Relative Entropies

The Shannon entropy of [0, 1] TFF(a, b, α, β) random variable X can be found as follows:

$$H = -\int_0^\infty f\,(x)\;\ln\big(f\,(x)\big)\;dx$$

$$= -\int_0^\infty f\,(x) \ln\left(\frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-ae^{\alpha b x^{-\beta}}}\right) dx$$

$$= \int_0^\infty f\,(x) \left[-\ln\left(\frac{ab\alpha\beta}{e^{-a}}\right) + (\beta+1)\ln(x) - \alpha b x^{-\beta} + a e^{\alpha b x^{-\beta}}\right] dx$$

$$= -\ln\left(\frac{ab\alpha\beta}{e^{-a}}\right) + (\beta+1)E(\ln(x)) - \alpha b E\left(x^{-\beta}\right) + aE\left(e^{\alpha b x^{-\beta}}\right).$$

Let $I_1 = (\beta+1)E(\ln(x))$

$$= (\beta+1)\int_0^\infty \ln(x)\frac{ab\alpha\beta}{e^{-a}}x^{-(\beta+1)}\left(e^{-\alpha x^{-\beta}}\right)^{-b}e^{-a\left(e^{-\alpha x^{-\beta}}\right)^{-b}}\,dx.$$

Since $e^{-a\left(e^{-\alpha x^{-\beta}}\right)^{-b}} = \sum_{u=0}^{\infty}\frac{(-1)^u}{u!}a^u\left(e^{-\alpha x^{-\beta}}\right)^{-bu}$, then

$$I_1 = (\beta+1)\frac{ab\alpha\beta}{e^{-a}}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}a^u\int_0^\infty \ln(x)\,x^{-(\beta+1)}\left(e^{-\alpha x^{-\beta}}\right)^{-b}\left(e^{-\alpha x^{-\beta}}\right)^{-bu}dx$$

$$= (\beta+1)\frac{ab\alpha\beta}{e^{-a}}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}a^u\int_0^\infty \ln(x)\,x^{-(\beta+1)}\left(e^{-\alpha x^{-\beta}}\right)^{-b(u+1)}\,dx.$$

Since $\left(e^{-\alpha x^{-\beta}}\right)^{-b(u+1)} = \left(1-\left(1-e^{-\alpha x^{-\beta}}\right)\right)^{-b(u+1)}$, by using

$$(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{j!\,\Gamma(k)}z^j \text{ and } (1-z)^{\delta} = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\frac{\Gamma(\delta+1)}{\Gamma(\delta-s+1)}\ |z|<1, \qquad k,\delta>0$$

is non-integer real number. We get,

$\left(1-\left(1-e^{-\alpha x^{-\beta}}\right)\right)^{-b(u+1)} = \sum_{j=0}^{\infty}\frac{\Gamma(b(u+1)+j)}{j!\Gamma(b(u+1))}\sum_{s=0}^{j}\frac{(-1)^s}{s!}\frac{\Gamma(j+1)}{\Gamma(j-s+1)}e^{-\alpha s x^{-\beta}}$, and then

$$I_1 = (\beta+1)\frac{ab\alpha\beta}{e^{-a}}\sum_{u=0}^{\infty}\frac{(-1)^{u+s}}{u!s!}a^u\sum_{j=0}^{\infty}\sum_{s=0}^{j}\frac{\Gamma(b(u+1)+j)}{j!\Gamma(b(u+1))}\ \frac{\Gamma(j+1)}{\Gamma(j-s+1)}\int_0^\infty \ln(x)\ x^{-(\beta+1)}\ e^{-\alpha s x^{-\beta}}\ dx\ .$$

Let $y = x^{-\beta} \Rightarrow x = y^{\frac{-1}{\beta}} \Rightarrow dx = \frac{-1}{\beta}y^{\frac{-1}{\beta}-1}dy$

$$I_1 = (\beta+1)\frac{ab\alpha\beta}{e^{-a}}\sum_{u=0}^{\infty}\frac{(-1)^{u+s}}{u!\,s!}a^u\sum_{j=0}^{\infty}\sum_{s=0}^{j}\frac{\Gamma(b(u+1)+j)}{j!\,\Gamma(b(u+1))}\ \frac{\Gamma(j+1)}{\Gamma(j-s+1)}\int_0^\infty \ln\left(y^{\frac{-1}{\beta}}\right)\ y^{1+\frac{1}{\beta}}\ e^{-\alpha s y}\ \frac{1}{\beta}\ y^{\frac{-1}{\beta}-1}\ dy$$

$$= -(\beta+1)\frac{b\alpha e^a}{\beta}\sum_{u=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\frac{(-1)^{u+s}}{u!\,s!}a^{u+1}\frac{\Gamma(b(u+1)+j)}{j!\,\Gamma(b(u+1))}\ \frac{\Gamma(j+1)}{\Gamma(j-s+1)}\int_0^\infty \ln(y)\ e^{-\alpha s y}\ dy\,.$$

Since $\int_0^\infty x^{s-1}\,e^{-mx}\ \ln(x)\ dx = m^{-s}\Gamma(s)\{\Psi(s)-\ln(m)\}$, where $\Psi(1) = -\gamma$ and $s = 1, m = \alpha s$

$$I_1 = -(\beta+1)\frac{b\alpha e^a}{\beta}\sum_{u=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\frac{(-1)^{u+s}}{u!\,s!}a^{u+1}\frac{\Gamma(b(u+1)+j)}{j!\,\Gamma(b(u+1))}\frac{\Gamma(j+1)}{\Gamma(j-s+1)}\ \frac{1}{\alpha s}\ \{-\gamma-\ln(\alpha s)\}$$

$$= (\beta+1)\frac{b e^a}{\beta s}\sum_{u=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\frac{(-1)^{u+s}}{u!\,s!}a^{u+1}\frac{\Gamma(b(u+1)+j)}{j!\,\Gamma(b(u+1))}\ \frac{\Gamma(j+1)}{\Gamma(j-s+1)}\{\gamma+\ln(\alpha s)\}\,,$$

and $I_2 = -\alpha b E\left(x^{-\beta}\right) = -\alpha b\int_0^\infty x^{-\beta}\frac{ab\alpha\beta}{e^{-a}}x^{-(\beta+1)}e^{\alpha b x^{-\beta}}e^{-a e^{\alpha b x^{-\beta}}}dx$

$$= -\frac{a\beta(\alpha b)^2}{e^{-a}}\int_0^\infty x^{-2\beta-1}\,e^{\alpha b x^{-\beta}}e^{-a e^{\alpha b x^{-\beta}}}dx.$$

Let $y = e^{\alpha b x^{-\beta}} \Rightarrow x = \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}} \Rightarrow dx = \frac{-1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1}\frac{1}{\alpha b y}dy$

$$I_2 = \frac{-a\beta(\alpha b)^2}{e^{-a}} \int_1^{\infty} \left(\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}}\right)^{-2\beta-1} y e^{-ay} \frac{1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b y} dy$$

$$= -ae^{a} \int_1^{\infty} \ln(y)\, e^{-ay} dy$$

$$= -ae^{a}\left[\int_0^{\infty} \ln(y)\, e^{-ay} dy - \int_0^{1} \ln(y)\, e^{-ay} dy\right]$$

$$I_{21} = \int_0^{\infty} \ln(y)\, e^{-ay} dy = \frac{1}{a}\{-\gamma - \ln(a)\} = \frac{-1}{a}\{\gamma + \ln(a)\}.$$

Since $\int_0^{\infty} x^{s-1} e^{-mx} \ln(x)\, dx = m^{-s}\Gamma(s)\{\Psi(s) - \ln(m)\}$, where $\Psi(1) = -\gamma$ and $s = 1, m = a$

$$I_{22} = \int_0^{1} \ln(y)\, e^{-ay} dy$$

$$= \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \int_0^{1} \ln(y)\, y^m dy = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} a^m}{m!\,(m+1)^2}.$$

Since $e^{-ay} = \sum_{m=0}^{\infty} \frac{(-ay)^m}{m!}$ and $\int x^m \ln(x)\, dx = x^{m+1}\left\{\frac{\ln(x)}{m+1} - \frac{1}{(m+1)^2}\right\}$, $I_2 = e^{a}\left\{\gamma + \ln(a) + \sum_{m=0}^{\infty} \frac{(-1)^{m+1} a^{m+1}}{m!(m+1)^2}\right\}$, and $I_3 = aE\left(e^{\alpha b x^{-\beta}}\right)$

$$= a \int_0^{\infty} e^{\alpha b x^{-\beta}} \frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-ae^{\alpha b x^{-\beta}}} dx.$$

Let $y = e^{\alpha b x^{-\beta}} \Rightarrow x = \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}} \Rightarrow dx = \frac{-1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b}\frac{1}{y} dy$, then

$$I_3 = \frac{a^2 b\alpha\beta}{e^{-a}} \int_1^{\infty} \left(\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}}\right)^{-(\beta+1)} y^2 e^{-ay} \frac{1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b}\frac{1}{y} dy$$

$$= a^2 e^{a} \int_1^{\infty} \left(\frac{\ln y}{\alpha b}\right)^{1+\frac{1}{\beta}} y e^{-ay} \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} dy$$

$$I_3 = a^2 e^{a} \int_1^{\infty} y\, e^{-ay} dy = a + 1$$

$$H = \ln\left(\frac{e^{-a}}{ab\alpha\beta}\right) + \frac{(\beta+1)be^{a}}{\beta s} \sum_{u=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{j} \frac{(-1)^{u+s}}{u!\,s!} a^{u+1} \frac{\Gamma(b(u+1)+j)}{j!\,\Gamma\big(b(u+1)\big)} \frac{\Gamma(j+1)}{\Gamma(j-s+1)}\{\gamma + \ln(\alpha s)\} + e^{a}\left\{\gamma + \ln(a) + \sum_{m=0}^{\infty} \frac{(-1)^{m+1} a^{m+1}}{m!\,(m+1)^2}\right\} + a + 1. \quad (28)$$

The relative entropy $Dkl(F\|F^*)$ for a random variable [0, 1] TFF(a, b, α, β) can be found as follows:

$$\frac{f(x)}{f^*(x)} = \frac{\frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}}}{\frac{a_1 b_1 \alpha_1 \beta_1}{e^{-a_1}} x^{-(\beta_1+1)} e^{\alpha_1 b_1 x^{-\beta_1}} e^{-a_1 e^{\alpha_1 b_1 x^{-\beta_1}}}}$$

$$Dkl(F_1 \| F_1^*) = \int_0^\infty f(x) \ln\left(\frac{ab\alpha\beta e^{-a_1} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}}}{a_1 b_1 \alpha_1 \beta_1 e^{-a} x^{-(\beta_1+1)} e^{\alpha_1 b_1 x^{-\beta_1}} e^{-a_1 e^{\alpha_1 b_1 x^{-\beta_1}}}}\right) dx$$

$$= \int_0^\infty f_1(x) \left[\begin{array}{c}\ln\left(\frac{ab\alpha\beta e^{-a_1}}{a_1 b_1 \alpha_1 \beta_1 e^{-a}}\right) + (\beta_1 - \beta)\ln(x) + \alpha b x^{-\beta} - a e^{\alpha b x^{-\beta}} - \\ \alpha_1 b_1 x^{-\beta_1} + a_1 e^{\alpha_1 b_1 x^{-\beta_1}}\end{array}\right] dx$$

$$= \ln\left(\frac{ab\alpha\beta e^{-a_1}}{a_1 b_1 \alpha_1 \beta_1 e^{-a}}\right) + (\beta_1 - \beta)E(\ln(x)) + \alpha b E\left(x^{-\beta}\right) - aE\left(e^{\alpha b x^{-\beta}}\right) - \alpha_1 b_1 E\left(x^{-\beta_1}\right) + a_1 E\left(e^{\alpha_1 b_1 x^{-\beta_1}}\right).$$

Let, $I_1 = (\beta_1 - \beta)E(\ln(x))$

$$= (\beta_1 - \beta)\frac{ab\alpha\beta}{e^{-a}} \int_0^\infty \ln(x)\, x^{-(\beta+1)} \left(e^{-\alpha x^{-\beta}}\right)^{-b} e^{-a\left(e^{-\alpha x^{-\beta}}\right)^{-b}} dx$$

$$I_1 = \frac{(\beta_1 - \beta) b e^a}{\beta s} \sum_{u=0}^\infty \sum_{j=0}^\infty \sum_{s=0}^j \frac{(-1)^{u+s}}{u!\, s!} a^{u+1} \frac{\Gamma(b(u+1)+j)}{j!\,\Gamma\big(b(u+1)\big)} \frac{\Gamma(j+1)}{\Gamma(j-s+1)} \{\gamma + \ln(\alpha s)\},$$

and $I_2 = \alpha b E\left(x^{-\beta}\right)$

$$= \alpha b \int_0^\infty x^{-\beta} \frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}} dx$$

$$I_2 = -e^a \left\{\gamma + \ln(a) + \sum_{m=0}^\infty \frac{(-1)^{m+1} a^{m+1}}{m!\,(m+1)^2}\right\},$$

and $I_3 = -aE\left(e^{\alpha b x^{-\beta}}\right)$

$$= -a \int_0^\infty e^{\alpha b x^{-\beta}} \frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}} dx$$

$$I_3 = -(a+1),$$

and $I_4 = -\alpha_1 b_1 E\left(x^{-\beta_1}\right)$

$$= -\alpha_1 b_1 \int_0^\infty x^{-\beta_1} \frac{ab\alpha\beta}{e^{-a}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}} dx.$$

Let $y = e^{\alpha b x^{-\beta}} \Rightarrow x = \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}} \Rightarrow dx = \frac{-1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b}\frac{1}{y} dy$

$$I_4 = \frac{-\alpha_1 b_1 ab\alpha\beta}{e^{-a}} \int_1^\infty \left(\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}}\right)^{-(\beta_1+\beta+1)} y e^{-ay} \frac{1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b}\frac{1}{y} dy$$

$$= \frac{-\alpha_1 b_1 a}{e^{-a}} \int_1^\infty \left(\frac{\ln y}{\alpha b}\right)^{\frac{\beta_1}{\beta}+1+\frac{1}{\beta}} e^{-ay} \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} dy$$

$$= -\alpha_1 b_1 a e^{a} (\alpha b)^{\frac{-\beta_1}{\beta}} \int_1^{\infty} (\ln y)^{\frac{\beta_1}{\beta}} e^{-ay} dy.$$

Since $e^{-ay} = \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} y^m$, then

$$I_4 = -\alpha_1 b_1 a e^{a} (\alpha b)^{\frac{-\beta_1}{\beta}} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \int_1^{\infty} (\ln y)^{\frac{\beta_1}{\beta}} y^m dy.$$

Let $y = e^{-t^{-1}} \Rightarrow t = (-\ln y)^{-1} \Rightarrow dy = t^{-2} e^{-t^{-1}} dt$, then

$$I_4 = \alpha_1 b_1 a e^{a} (\alpha b)^{\frac{-\beta_1}{\beta}} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \int_0^{\infty} \left(\ln e^{-t^{-1}}\right)^{\frac{\beta_1}{\beta}} \left(e^{-t^{-1}}\right)^m t^{-2} e^{-t^{-1}} dt$$

$$= \alpha_1 b_1 a e^{a} (\alpha b)^{\frac{-\beta_1}{\beta}} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} (-1)^{\frac{\beta_1}{\beta}} \int_0^{\infty} t^{\frac{-\beta_1}{\beta}} t^{-2} e^{-(m+1)t^{-1}} dt.$$

Let $w = t^{-1} \Rightarrow t = w^{-1} \Rightarrow dt = -w^{-2} dw$

$$I_4 = \alpha_1 b_1 a e^{a} (\alpha b)^{\frac{-\beta_1}{\beta}} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} (-1)^{\frac{\beta_1}{\beta}} \int_0^{\infty} w^{\frac{\beta_1}{\beta}} w^2 e^{-(m+1)w} w^{-2} dw$$

$$= \alpha_1 b_1 a e^{a} (\alpha b)^{\frac{-\beta_1}{\beta}} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} (-1)^{\frac{\beta_1}{\beta}} \int_0^{\infty} w^{\frac{\beta_1}{\beta}} e^{-(m+1)w} dw$$

$$= \alpha_1 b_1 e^{a} (\alpha b)^{\frac{-\beta_1}{\beta}} \sum_{m=0}^{\infty} \frac{(-1)^{m+\frac{\beta_1}{\beta}}}{m!\,(m+1)^{\frac{\beta_1}{\beta}+1}} a^{m+1} \Gamma\left(\frac{\beta_1}{\beta}+1\right),$$

and $I_5 = a_1 E\left(e^{\alpha_1 b_1 x^{-\beta_1}}\right)$

$$= \frac{a_1 a b \alpha \beta}{e^{-a}} \int_0^{\infty} e^{\alpha_1 b_1 x^{-\beta_1}} x^{-(\beta+1)} e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}} dx.$$

Since $e^{\alpha_1 b_1 x^{-\beta_1}} = \sum_{s=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} x^{-\beta_1 s}$, then

$$I_5 = \frac{a_1 a b \alpha \beta}{e^{-a}} \sum_{s=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \int_0^{\infty} x^{-\beta_1 s - \beta - 1} e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}} dx.$$

Let $y = e^{\alpha b x^{-\beta}} \Rightarrow x = \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}} \Rightarrow dx = \frac{-1}{\beta} \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b y} dy$

$$I_5 = \frac{a_1 a b \alpha \beta}{e^{-a}} \sum_{s=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \int_1^{\infty} \left(\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}}\right)^{-\beta_1 s - \beta - 1} y e^{-ay} \frac{1}{\beta} \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b} \frac{1}{y} dy$$

$$= a_1 a e^{a} \sum_{s=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \int_1^{\infty} \left(\frac{\ln y}{\alpha b}\right)^{\frac{\beta_1 s}{\beta}} e^{-ay} dy.$$

So, by using, $e^{-ay} = \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} y^m$, we get

$$I_5 = a_1 a e^{a} \sum_{s=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \int_1^{\infty} \left(\frac{\ln y}{\alpha b}\right)^{\frac{\beta_1 s}{\beta}} y^m dy.$$

Let $y = e^{-t^{-1}} \Rightarrow t = (-\ln y)^{-1} \Rightarrow dy = t^{-2}e^{-t^{-1}}dt$

$$I_5 = -a_1 a e^a \sum_{s=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \int_0^{\infty} \left(\frac{\ln e^{-t^{-1}}}{\alpha b}\right)^{\frac{\beta_1 s}{\beta}} \left(e^{-t^{-1}}\right)^m t^{-2} e^{-t^{-1}} dt$$

$$= -a_1 e^a \sum_{s=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^{m+1} (-1)^{\frac{\beta_1 s}{\beta}} (\alpha b)^{\frac{-\beta_1 s}{\beta}} \int_0^{\infty} t^{\frac{-\beta_1 s}{\beta}} e^{-(m+1)t^{-1}} t^{-2} dt.$$

Let $w = t^{-1} \Rightarrow t = w^{-1} \Rightarrow dt = -w^{-2}dw$

$$I_5 = -a_1 e^a \sum_{s,m=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \frac{(-1)^{m+\frac{\beta_1 s}{\beta}}}{m!} a^{m+1} (\alpha b)^{\frac{-\beta_1 s}{\beta}} \int_0^{\infty} w^{\frac{\beta_1 s}{\beta}} e^{-(m+1)w} dw$$

$$= -a_1 e^a \sum_{s,m=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \frac{(-1)^{m+\frac{\beta_1 s}{\beta}}}{m!} a^{m+1} (\alpha b)^{\frac{-\beta_1 s}{\beta}} \frac{\Gamma\left(\frac{\beta_1 s}{\beta} + 1\right)}{(m+1)^{\frac{\beta_1 s}{\beta}+1}}$$

$$Dkl(F_1 \| F_1^*) = \ln\left(\frac{ab\alpha\beta e^{-a_1}}{a_1 b_1 \alpha_1 \beta_1 e^{-a}}\right) + (\beta_1 - \beta) \frac{be^a}{\beta s} \sum_{u=0}^{\infty} \sum_{j=0}^{\infty} \sum_{s=0}^{j} \frac{(-1)^{u+s}}{u!\, s!} a^{u+1} \frac{\Gamma(b(u+1)+j)}{j!\, \Gamma\big(b(u+1)\big)}$$
$$\frac{\Gamma(j+1)}{\Gamma(j-s+1)} \{\gamma + \ln(\alpha s)\} - e^a \left\{\gamma + \ln(a) + \sum_{m=0}^{\infty} \frac{(-1)^{m+1} a^{m+1}}{m!\,(m+1)^2}\right\} - (a+1) + \alpha_1 b_1 e^a$$
$$(\alpha b)^{\frac{-\beta_1}{\beta}} \sum_{m=0}^{\infty} \frac{(-1)^{m+\frac{\beta_1}{\beta}}}{m!} a^{m+1} \frac{\Gamma\left(\frac{\beta_1}{\beta} + 1\right)}{(m+1)^{\frac{\beta_1}{\beta}+1}} \tag{29}$$
$$- a_1 e^a \sum_{s,m=0}^{\infty} \frac{(\alpha_1 b_1)^s}{s!} \frac{(-1)^{m+\frac{\beta_1 s}{\beta}}}{m!} a^{m+1} (\alpha b)^{\frac{-\beta_1 s}{\beta}} \frac{\Gamma\left(\frac{\beta_1 s}{\beta} + 1\right)}{(m+1)^{\frac{\beta_1 s}{\beta}+1}}.$$

3.2. Stress-Strength Reliability

Let y and x be the stress and the strength random variables, independent of each other, follow [0, 1] TFF(a, b, α, β) and[0, 1]TFF$(a_1, b_1, \alpha_1, \beta_1)$, respectively. Then

$$R = P(Y < X) = \int_0^{\infty} f_x(x)\, F_y(x)\, dx$$

$$= \int_0^{\infty} \frac{ab\alpha\beta}{e^{-a}}\, x^{-(\beta+1)}\, e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}}\, \frac{e^{-a_1 e^{\alpha_1 b_1 x^{-\beta_1}}}}{e^{-a_1}} dx.$$

Since $e^{-a_1 e^{\alpha_1 b_1 x^{-\beta_1}}} = \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-a_1)^i}{i!} \frac{(i\alpha_1 b_1)^s}{s!} x^{-\beta_1 s}$, then

$$R = \frac{ab\alpha\beta}{e^{-(a+a_1)}} \sum_{i,s=0}^{\infty} \frac{(-a_1)^i}{i!} \frac{(i\alpha_1 b_1)^s}{s!} \int_0^{\infty} x^{-\beta_1 s - \beta - 1}\, e^{\alpha b x^{-\beta}} e^{-a e^{\alpha b x^{-\beta}}} dx.$$

Let $y = e^{\alpha b x^{-\beta}} \Rightarrow x = \left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}} \Rightarrow dx = \frac{-1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b}\frac{1}{y}dy$

$$R = \frac{ab\alpha\beta}{e^{-(a+a_1)}} \sum_{i,s=0}^{\infty} \frac{(-a_1)^i}{i!}\frac{(i\alpha_1 b_1)^s}{s!} \int_1^{\infty} \left(\frac{\ln y}{\alpha b}\right)^{\frac{\beta_1 s}{\beta}+1+\frac{1}{\beta}} y e^{-ay} \frac{1}{\beta}\left(\frac{\ln y}{\alpha b}\right)^{\frac{-1}{\beta}-1} \frac{1}{\alpha b}\frac{1}{y} dy$$

$$= a e^{a+a_1} \sum_{i,s=0}^{\infty} \frac{(-a_1)^i}{i!}\frac{(i\alpha_1 b_1)^s}{s!} \int_1^{\infty} \left(\frac{\ln y}{\alpha b}\right)^{\frac{\beta_1 s}{\beta}} e^{-ay} dy.$$

So, by using $e^{-ay} = \sum_{m=0}^{\infty}\frac{(-a)^m}{m!}y^m$, we get

$$R = a e^{a+a_1} \sum_{i,s=0}^{\infty} \frac{(-a_1)^i}{i!}\frac{(i\alpha_1 b_1)^s}{s!} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} (\alpha b)^{\frac{-\beta_1 s}{\beta}} \int_1^{\infty} (\ln y)^{\frac{\beta_1 s}{\beta}} y^m dy.$$

Let $y = e^{-t^{-1}} \Rightarrow t = (-\ln y)^{-1} \Rightarrow dy = t^{-2}e^{-t^{-1}}dt$

$$R = -a e^{a+a_1} \sum_{i,s=0}^{\infty} \frac{(-a_1)^i}{i!}\frac{(i\alpha_1 b_1)^s}{s!} \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} (\alpha b)^{\frac{-\beta_1 s}{\beta}} \int_0^{\infty} \left(\ln e^{-t^{-1}}\right)^{\frac{\beta_1 s}{\beta}} \left(e^{-t^{-1}}\right)^m t^{-2} e^{-t^{-1}} dt$$

$$= -e^{a+a_1} \sum_{i,s,m=0}^{\infty} \frac{(-1)^{i+m+\frac{\beta_1 s}{\beta}}}{i!\,m!} a_1^i \frac{\left(i\alpha_1 b_1 (\alpha b)^{-\beta_1/\beta}\right)^s}{s!} a^{m+1} \int_0^{\infty} t^{\frac{-\beta_1 s}{\beta}} e^{-(m+1)t^{-1}} t^{-2} dt.$$

Let $w = t^{-1} \Rightarrow t = w^{-1} \Rightarrow dt = -w^{-2}dw$

$$R = -e^{a+a_1} \sum_{i,s,m=0}^{\infty} \frac{(-1)^{i+m+\frac{\beta_1 s}{\beta}}}{i!\,m!} a_1^i \frac{\left(i\alpha_1 b_1 (\alpha b)^{-\beta_1/\beta}\right)^s}{s!} a^{m+1} \int_0^{\infty} w^{\frac{\beta_1 s}{\beta}} e^{-(m+1)w} w^2 w^{-2} dw$$

$$= -e^{a+a_1} \sum_{i,s,m=0}^{\infty} \frac{(-1)^{i+m+\frac{\beta_1 s}{\beta}}}{i!\,m!} a_1^i \frac{\left(i\alpha_1 b_1 (\alpha b)^{-\beta_1/\beta}\right)^s}{s!} a^{m+1} \int_0^{\infty} w^{\frac{\beta_1 s}{\beta}} e^{-(m+1)w} dw$$

$$R = -e^{a+a_1} \sum_{i,s,m=0}^{\infty} \frac{(-1)^{i+m+\frac{\beta_1 s}{\beta}}}{i!\,m!} a_1^i \frac{\left(i\alpha_1 b_1 (\alpha b)^{-\beta_1/\beta}\right)^s}{s!} a^{m+1} \frac{\Gamma\left(\frac{\beta_1 s}{\beta}+1\right)}{(m+1)^{\frac{\beta_1 s}{\beta}+1}}.$$

4. Summary and Conclusions

In statistical analysis, a lot of distributions are used to represent set(s) data. Recently, new distributions are derived to extend some of well-known families of distributions; such the new distributions are more flexible than the others to model real data are. The composing of some distributions with each other's some way has been in the foreword of data modeling. In this paper, we presented a new family of continuous distributions based on [0, 1] truncated Fréchet distribution. [0, 1] Truncated Fréchet-Weibull ([0, 1] TFU) and [0, 1] Truncated Fréchet- Fréchet ([0,1] TFF) distributions that were discussed as special cases. Properties of [0, 1] TFW and [0, 1] TFF was derived. We provided forms for characteristic function, rth raw moment, mean, variance, skewness, kurtosis, median, reliability function, hazard rate function, Shannon entropy function, and Relative entropy function. This paper also dealt with the determination of stress strength reliability R = P[Y < X] when X (strength) and Y (stress) were two independent [0, 1] TFW ([0, 1] TFF) distributions with different parameters.

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