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Non-Standard Finite Difference Schemes for Solving Singular Lane-Emden Equation

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ABSTRACT

In this paper we construct Non-Standard finite difference schemes (NSFD) for numerical solution of nonlinear Lane-Emden type equations which are nonlinear ordinary dierential equations on semi-infinite domain. They are categorized as singular initial value problems. This equation describes a variety of phenomena in theoretical physics and astrophysics. The presented schemes are obtained by using the Non-Standard finite difference method. The use of NSFD method and its approximations play an important role for the formation of stable numerical methods. The main advantage of the schemes is that the algorithm is very simple and very easy to implement. Thus, this method may be applied as a simple and accurate solver for ODEs and PDEs and it can also be utilized as an accurate algorithm to solve linear and nonlinear equations arising in physics and other fields of applied mathematics. Illustrative examples have been discussed to demonstrate validity and applicability of the technique and the results have been compared with the exact solutions.

Keywords: *Non-Standard finite, difference, Lane-Emden equation, A*strophysics.

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1. Introduction

Lane–Emden type equations have significant applications in many fields of scientific and technical world. One of the important fields of application of the Lane-Emden equation is the analysis of the diffusive transport and chemical reaction of species inside a porous catalyst particle. These equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytrophic theory of stars essentially follows out of thermodynamic considerations that deal with the issue of energy transport, through the transfer of material between different levels of

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the star. This equation is one of the basic equations in the theory of stellar structure. The general form of the Lane-Emden equation is as follows:

$$y''(x)+\frac{m}{x}y'(x)+f(x,y)=g(x), \quad 0\prec x\le 1, \quad m\ge 0, \tag{1}$$

with the following initial conditions:

$$y(0)=A, \quad y'(0)=B. \tag{2}$$

$f(x,y)$ is a continuous real-value function and $g(x)$ is an analytical function. This equation has been the focus of many studies. H. Aaminikhah and S. Moradian [1] solved Lane-Emden type equations by the Legendre wavelet method. Liao [2] solved this equation by applying a homotopy analysis method. He [3] obtained an approximate analytical solution of the Lane-Emden equation by applying a variational approach. Bender et al. [4] proposed a new perturbation technique. Shawagfeh [5] and Wazwaz [6, 7] solve this equation by applying the Adomian method which provides a convergent series solution. Ramos [8] presented a series approach to the Lane-Emden equation. Parand et al. [9–12] presented three numerical techniques to solve higher ordinary differential equations such as Lane-Emden. El-Gebeily and O'Regan [13] used the quasilinearization approach to solve the standard Lane-Emden equation. Mandelzweig and Tabakin [14] applied Bellman and Kalaba's quasilinearization method, and Ramos [15] used a piecewise linearization technique based on the piecewise linearization of the Lane-Emden equation. Bozkhov and Gilli Martins [16] and later Momoniat and Harley [17] applied the Lie Group method to generalized Lane-Emden equations of the first kind. Özis and Yildirim [18, 19] gave the solutions of a class of singular second-order IVPs of Lane-Emden type by using homotopy perturbation and variational iteration method. In this paper, the new NSFD schemes will be introduced for numerical solutions of Lane-Emden type equations.

2. Non-Standard Finite Difference Method

The fundamental of the Non-Standard finite difference method was first developed by Ronald Mickens [20-24]. The NSFD schemes have been used as a good method for solving many problems [25-29]. This method is based on two rules [22]:

i. The discrete first-order derivative must take a more general form than that used in standard discretization, i.e.,

$$\frac{dy}{dx}\to\frac{y_{k+1}-\psi(h)y_k}{\phi(h)}, \tag{3}$$

where $\psi(h)$ and $\phi(h)$ are known, respectively, as the numerator and denominator functions, having the properties

$$\psi(h) = 1 + O(h), \quad \phi(h) = h + O(h^2), \tag{4}$$

where $h = \Delta x,\ x \to x_k = hk,$ and $y(x) \to y_k$. The second-order derivative discrete in the following form.

$$\frac{d^2 y}{dx^2} \to \frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi(h)}, \tag{5}$$

where

$$\varphi(h) = h^2 + O(h^4). \tag{6}$$

ii. Both linear and nonlinear terms involving the dependent variable may require

" nonlocal " discretization; for example

$$y = 2y - y \to 2y_k - y_{k+1},$$

$$y^2 = yy \to y_k y_{k+1}, \tag{7}$$

$$y^3 = \frac{(y+y)}{2} y^2 \to \frac{(y_{k+1} + y_{k-1})}{2} y_k^2.$$

The full details of these procedures are given in [20-24]. In this work we apply NSFD schemes for solving Lane-Emden type equations.

3. Numerical Results

In this section we construct the NSFD schemes to obtain numerical solution for Lane-Emden type equations.

Example 1. Consider the following nonlinear Lane-Emden type equation.

$$y''(x) + \frac{2}{x} y'(x) + y^n(x) = 0, \quad 0 \prec x \le 1, \tag{8}$$

Subject to the initial conditions:

$$y(0)=1,\ y^{'}(0)=0, \tag{9}$$

where $n \geq 0$ is constant. Substituting $n=0,1$ and 5 in to (8) leads to the exact solution

$$y(x)=1-\frac{1}{3!}x^{2},$$
$$y(x)=\frac{\sin(x)}{x}, \tag{10}$$
$$y(x)=\frac{1}{\sqrt{1+\frac{x^{2}}{3}}}.$$

For $n=0$ we have

$$y^{''}(x)+\frac{2}{x}y^{'}(x)+1=0,\ y(0)=1,\ y^{'}(0)=0,\ 0\prec x\leq 1, \tag{11}$$

For solving (11) we construct the following NSFD scheme.

$$\frac{y_{k+1}-2y_{k}+y_{k-1}}{4\sin^{2}(\frac{h}{2})}+\frac{2}{x_{k}}\left(\frac{y_{k+1}-y_{k}}{\frac{1-e^{-3h}}{3}}\right)+1=0. \tag{12}$$

After some manipulation we have

$$y_{k+1}=\frac{\left(2x_{k}(1-e^{-3h})+24\sin^{2}(\frac{h}{2})\right)y_{k}-x_{k}(1-e^{-3h})y_{k-1}-4x_{k}(1-e^{-3h})\sin^{2}(\frac{h}{2})}{x_{k}(1-e^{-3h})+24\sin^{2}(\frac{h}{2})}. \tag{13}$$

In this example the denominator functions for first and second order derivative are

$$\phi(h)=\frac{1-e^{-3h}}{3},\ \varphi(h)=4\sin^{2}(\frac{h}{2}), \tag{14}$$

Respectively, note that $\phi(h)$ and $\varphi(h)$ satisfied relations (4) and (6). The numerator function chose as $\psi(h)=1$. In figure 1 the results of scheme (13) is compared with the exact solution.

For $n=1$ we have

$$y^{''}+\frac{2}{x}y^{'}+y=0,\ y(0)=1,\ y'(0)=0,\ 0\prec x\leq 1, \tag{15}$$

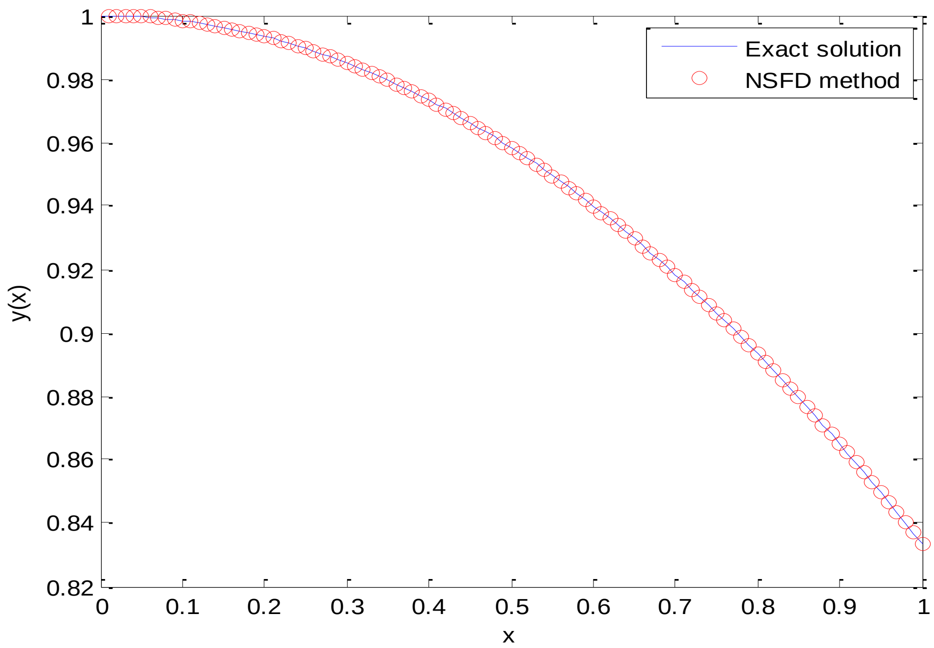

Fig 1. The results of NSFD scheme (13) with h=0.01 and exact solution of equation (11).

In this example we choose denominator functions as follows.

$$\varphi(h)=4\sin^2\left(\frac{h}{2}\right),\quad \phi(h)=e^h-1\ , \tag{16}$$

Therefore, we have the following NSFD scheme for solving (15).

$$y_{k+1}=\frac{\left(2x_k(e^h-1)+8\sin^2\left(\frac{h}{2}\right)-4x_k(e^h-1)\sin^2\left(\frac{h}{2}\right)\right)y_k-x_k(e^h-1)y_{k-1}}{x_k(e^h-1)+8\sin^2\left(\frac{h}{2}\right)}. \tag{17}$$

In Fig. 2 the result of scheme (17) is compared with the exact solution. For $n=5$ we have

$$y''+\frac{2}{x}y'+y^5=0,\qquad 0\prec x\le 1,\ \ y(0)=1,\ \ y'(0)=0\ . \tag{18}$$

For solving (18) we construct the following NSFD scheme.

$$\frac{y_{k+1}-2y_k+y_{k-1}}{4\sin^2\left(\frac{h}{2}\right)}+\frac{2}{x_k}\left(\frac{y_{k+1}-y_k}{e^h-1}\right)+y_{k+1}y_k^4=0. \tag{19}$$

Solving (19) in y_{k+1} yield:

$$y_{k+1}=\frac{\left(2x_k(e^h-1)+8\sin^2(\frac{h}{2})\right)y_k-x_k(e^h-1)y_{k-1}}{x_k(e^h-1)+8\sin^2(\frac{h}{2})+4x_k(e^h-1)\sin^2(\frac{h}{2})y_k^4}. \tag{20}$$

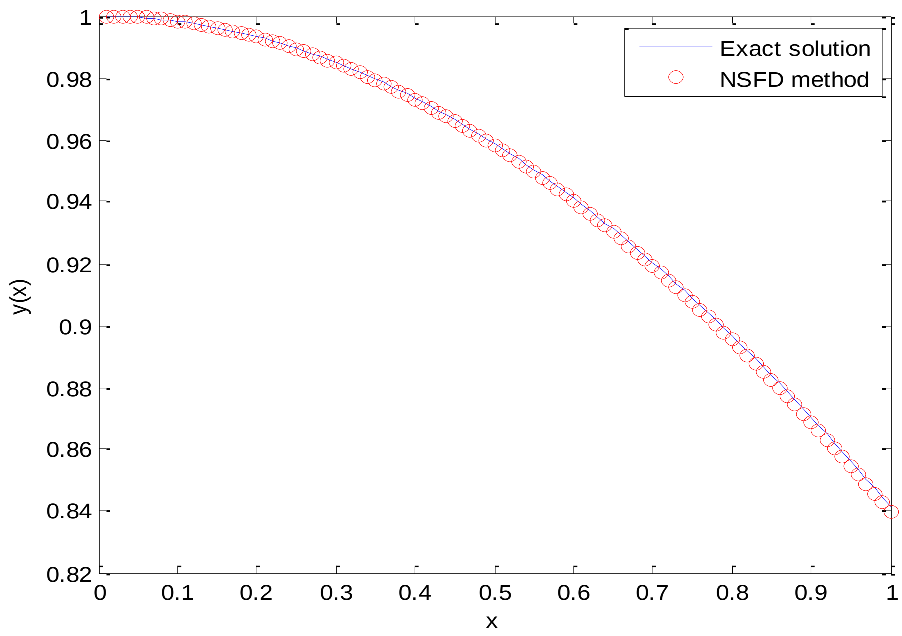

Fig 2. The results of NSFD scheme (17) with h=0.01 and exact solution of equation (15).

In Fig. 3 the result of scheme (20) was compared with the exact solution.

Example 2. Consider the following nonlinear Lane-Emden type equation.

$$y''+\frac{8}{x}y'+xy=x^5-x^4+44x^2-30x,\quad 0\prec x\le 1,\quad y(0)=0,\quad y'(0)=0. \tag{21}$$

The exact solution of (21) is

$$y(x)=x^4-x^3. \tag{22}$$

For solving (21) we have

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi(h)}+\frac{2}{x_k}\left(\frac{y_{k+1}-y_k}{\phi(h)}\right)+x_k\left(\frac{y_{k+1}+y_{k-1}}{2}\right)=x_k^5-x_k^4+44x_k^2-30x_k\,. \tag{23}$$

Therefore

$$y_{k+1}=\frac{\left(2x_k\phi+8\varphi\right)y_k-(\frac{1}{2}\phi\varphi x_k^2+x_k\phi)y_{k-1}+\phi\varphi(x_k^6-x_k^5+44x_k^3-30x_k^2)}{\frac{1}{2}\phi\varphi x_k^2+\phi x_k+8\varphi}. \tag{24}$$

In this case we choose $\phi(h)$ and $\varphi(h)$ as follows:

$$\phi(h)=e^h-1,\ \varphi(h)=4\sin^2(\frac{h}{2}). \tag{25}$$

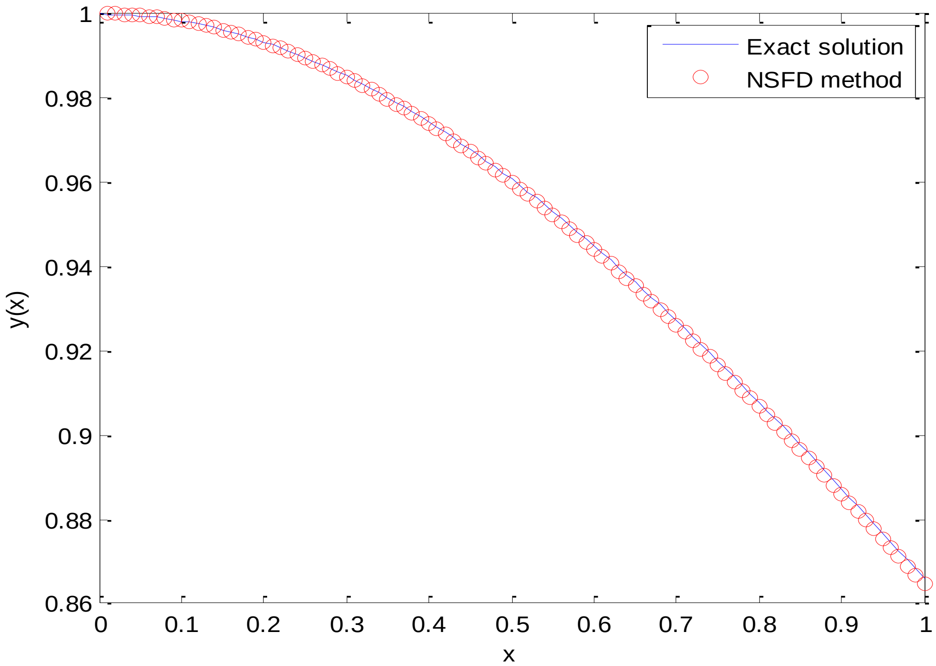

Fig 3. The results of NSFD scheme (20) with h=0.01 and exact solution of equation (18).

In Fig. 4 the result of scheme (24) is compared with the exact solution.

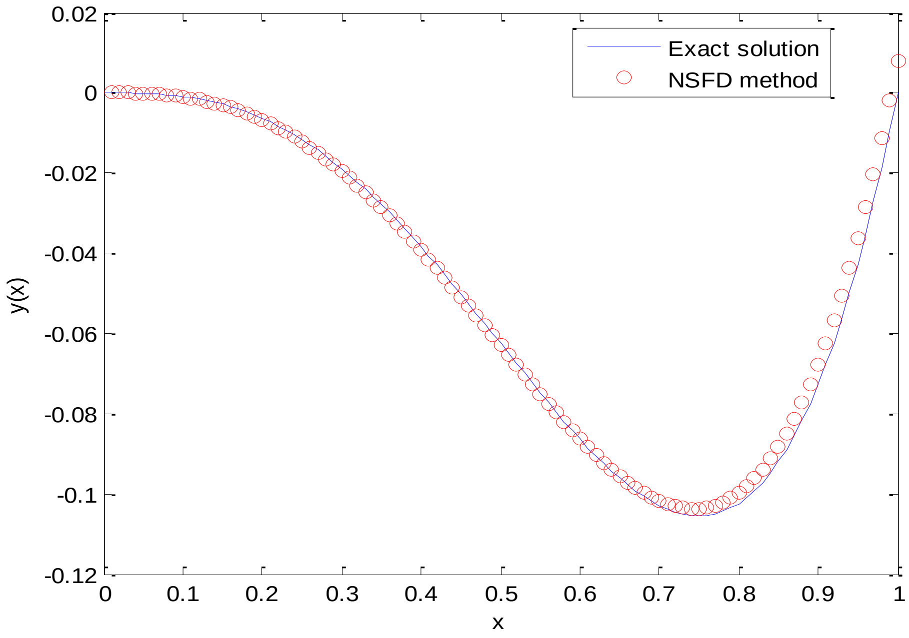

Fig 4. The results of NSFD scheme (24) with h=0.01 and Exact solution of equation (21).

Example 3. Consider the following nonlinear Lane-Emden type equation

$$y'' + \frac{6}{x} y' + 14y = -4y \ln y, \quad 0 \prec x \leq 1, \quad y(0) = 1, \quad y'(0) = 0. \tag{26}$$

The exact solution of (26) is

$$y(x) = e^{-x^2}. \tag{27}$$

For solving (26) by NSFD method we have

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi(h)} + \frac{6}{x_k}\left(\frac{y_{k+1} - y_k}{\phi(h)}\right) + 14y_k + 4y_k \ln y_k = 0 \tag{28}$$

Therefore

$$y_{k+1} = \frac{(2x_k\phi + 6\varphi - 14x_k\phi\varphi - 4x_k\phi\varphi \ln y_k)y_k - x_k\phi y_{k-1}}{x_k\phi + 6\varphi}, \tag{29}$$

where $\phi(h)$ and $\varphi(h)$ are

$$\phi(h) = e^h - 1,\ \varphi(h) = 4\sin^2(\frac{h}{2}). \tag{30}$$

In Fig.5 the result of scheme (29) is compared with the exact solution.

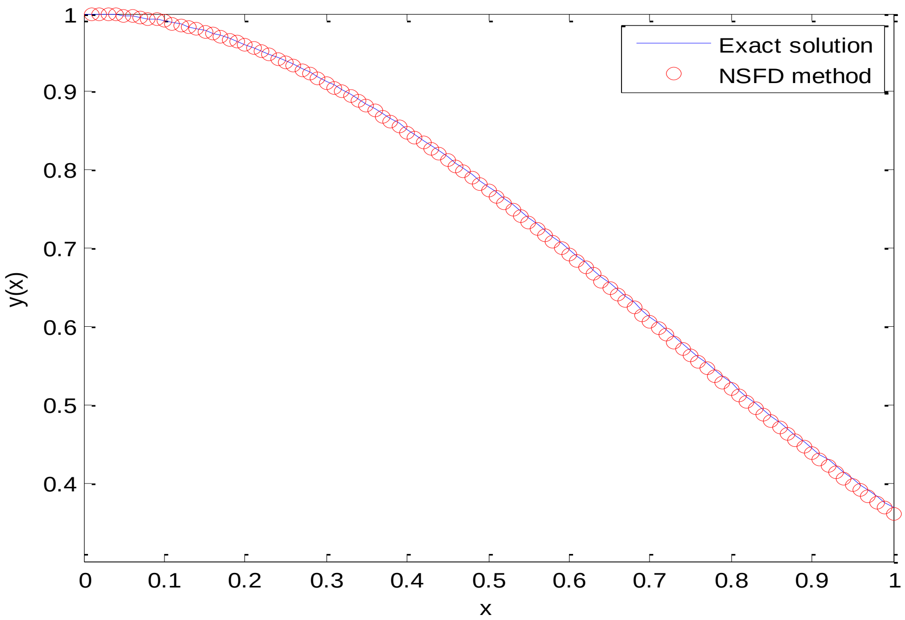

Figure 5. The results of NSFD scheme (29) with h=0.01 and Exact solution of equation (26).

4. Conclusion

In this paper we have presented Non-Standard finite difference schemes for numerical solution of Lane-Emden equation which is a second order nonlinear ODE. These schemes are explicit. Our results are compared with the exact solutions. From the graphical results in figures, it is clear that the approximate solutions are in good agreement with the exact solutions.

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