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Fuzzy Index Tracking Portfolio Selection Model Based on Value-at-Risk

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A B S T R A C T

Index tracking is one of the most important passive strategies which describes the process of attempting to track the performance of some specified benchmark indexes. Most recent studies determined security returns in conventional models by the precise historical data. However, such precise data are not always available and it is hard to forecast security returns with stochastic values. Therefore, to handle such imprecise uncertainty, considering security returns as variables with imprecise distributions, i.e., fuzzy variables are recommended. In these studies, researchers have studied and experimented with various risk-measure methods for index tracking portfolio selection. Models which were extended based on Markowitz portfolio selection model have used the single period variance of returns as a risk measure. Since forecasting future returns of portfolio is uncertain, we consider these returns as fuzzy variables in this study. We also apply Value-at-Risk as the risk measure whichhas not yet been established as risk measure in index tracking portfolio selection problems. The model is tested, using Tehran Price Index (TEPIX) and computational results are presented at the end.

1. Introduction

For many years, fund managers were finding the best strategies to provide a combination of capital growth and income. They dividedthese strategies in two classes: (a) *Active management*: in this strategy, investors carry out securities exchange actively so that they can find profit opportunity constantly. The assumption underlying this strategy is that fund managers can, through their expertise and judgment, add value more than passive strategy though choosing high performing stocks and/or by the timing of their buy and sell decisions. (b) *Passive management*: in this strategy, investors consider that the securities market is efficient. Therefore they cannot go beyond the average level of market permanently. Hence, obtaining approximately the same return as market return can satisfy them. This strategy emphasizesto minimize transaction costs.

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Index tracking is one of the most important passive strategies which describe the process of attempting to track the performance of some specified benchmark indexes. A simple way to track an index is to design a portfolio which holds all stocks in the market with the same relative quantities. But this method increases transaction costs and needs frequent revisions. So, index tracking portfolio selection models were extended in 1970s to track the performance of an index with specified number of stocks.

In recent years, researchers have studied and experimented with various risk-measure methods for index tracking portfolio selection. Models which were extended based on Markowitz portfolio selection model have used the single period variance of returns as a risk measure. From then on, various risk methods have been proposed. One of these methods is the "value-at-risk" (VaR). Nevertheless, index tracking portfolio selection model with VaR have not yet been established. This is because conventional stochastic VaR theory is not applicable to the portfolio selection problems in fuzzy environments. Wang et al. [1] proposed a proper definition of the fuzzy VaR.Since considering a specific number for a portfolio as its future return is uncertain, we used fuzzy variables.

The rest of the paper is organized as follows. In the next section, we review the literature of the problem. We represent index tracking portfolio selection model in section three. In section four, some concepts about fuzzy variables and a survey of fuzzy distance will be introduced. In addition, inthis section, fuzzy Value-at-Risk in some theorems will be represented and fuzzy index tracking portfolio selection model based on Value-at-Risk is illustrated. In section five,we test this model by TSE (Tehran Stock Exchange) market data (TEPIX) and make TSE index tracking portfolio based on fuzzy returns. Finally, the concluding remarks come in section six.

2. Literature review

Many researches have been studied on the Index tracking. These researches can be classified in three classes. Consist of Markowitz models, factor based models and other models.

2.1. Markowitz models

These models use Markowitz standard mean-variance model. In fact the objective of these models is to minimize tracking error variance. Holdges[2] was the first who used this model to compare the tradeoff curve relating variance to return in excess of the index with the tradeoff curve for the standard Markowitz portfolio optimization model. Roll [3] combined Markowitz and factor models. Rohweder[4] included transaction costs in his model. Wang [5]presented a model in which the objective function tracks more than one index. Beasley et al. [6] used Markowitz model which include transaction cost and presented an evolutionary heuristic method to solve this model.

2.2. Factor based models

These models relate the stock returns to one or more economic factors. In fact, these models regresses stock return on one or more factors like market return, GDP, inflation, etc. Rudd [7] presented a single factor model. He also included transaction costs in his model and solved it with a heuristic method and finally, the application was tested on portfolios tracking S&P 500 index.Corielli and Marcellino [8]represented index components with a dynamic factor model. They developed a procedure which in first step, builds a portfolio that is driven by the same persistent factors as the index. In the second step, it is also possible to refine the portfolio in order to minimize a specific loss function.

2.3. Some other works

Since solution space is extensive, methods which determine exact optimum solution, lose their efficiency. To cover this problem, researchers applied heuristics and metaheuristic methods to solve models. Recent studies show that applying these methods have been common to solve complex models.Okay [9] used constraint aggregation approach to solve his model. This method changes constraints of a large problem to one or less, in order to miniaturize the dimension of the problem. He showed that both common optimization method and constraint aggregation approach bring same solution. Constraint aggregation method reduced solution time. Oh et al. [10] applied genetic algorithm to solve the problem of South Korea stock market index tracking portfolio selection.Frino et al. [11] focused on improving index tracking. For this purpose, they used Australia stock market data. Krink et al. [12] compared some metaheuristic approach in solving index tacking problem. They compared the results of differentialevolution, genetic algorithm, simulated annealing and partial swarm optimization approaches and expressed that in complex problems, differential evolution method shows the best performance. Torrubiano and Alberto [13] introduced the combination of evolution algorithms and quadratic programming in his study as a hybrid strategy to solve index tracking problems. He asserted that his approach decreases solution time.

All the above studies determined security returns in conventional models by the precise historical data. However, such precise data are not always availableand it is hard to forecast security returns with stochastic values. Therefore, to handle such imprecise uncertainty,considering security returns as variables with imprecise distributions, i.e., fuzzy variables are recommended [14].Watada [15]applied fuzzy set theory inportfolio selection models. He extended Markowitz’s mean-variance idea to the fuzzy environment. Ghazanfar-Ahari et al. [16] had applied a fuzzy AHP method to allocate a limited fund among Pharmaceutical industry in Tehran Stock Exchange. Alimi et al. [17] applied a multi-objective fuzzy mean-semivariance model to determine the optimum portfolio.

3. Index Tracking model

In this section we define the problem of index tracking more precisely. It will be shown that this problem leads to a family of optimization problems which are defined by a particular choice of the distance or risk measures.

3.1. Notation

- n : is the total number of stocks in which we can invest
- K : is the desired number of stocks in the tracking portfolio
- x_i : is the proportion of stock i in the tracking portfolio
- x^l : is the minimum proportion of portfolio which must be held in stock i
- x^u :is the maximum proportion of portfolio which must be held in stock i
- P_i^t : is the price of stock i at time t
- r_i^t : is the return of stock i at time t

Where $r_i^t = \ln \frac{P_i^t}{P_i^{t-1}}$ (1)

- R_p : is the return of tracking portfolio

Where $R_p^t = \sum_{i=1}^{n} x_i r_i^t$ (2)

- μ^t : is the index value at time t
- I : is the return of index

Where $I^t = \ln \frac{\mu^t}{\mu^{t-1}}$ (3)

- z_i : is a binary variable and it takes 1 if any of stock i is held in the tracking portfolio; otherwise it takes 0.

3.2.Model specification

As mentioned before, index tracking problem describes the process of attempting to track the performance of some specified benchmark indexes. To reduce transaction costs, only special number of stocks which constitute index, should be held in portfolio. In other words, we're going to construct a portfolio with K stock to track the index precisely. So, the objective function will be minimizing the tracking error which is the difference between portfolio return and index return. Hence, we can formulate this problem as below:

$$\textit{Minimize} \quad Z = F(R_p, I) \tag{4}$$

Subject to

$$x^l z_i \le x_i \le x^u z_i \ , \ \forall \mathrm{i} \tag{5}$$

$$\sum_{i=1}^{n} z_i = K \tag{6}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{7}$$

$$x_\mathrm{i} \ge 0 \ , \ \forall \mathrm{i} \tag{8}$$

$$z_\mathrm{i} \in \{0,1\} \ , \ \forall \mathrm{i} \tag{9}$$

Eq. (4) is the objective function and represents that the model is going to minimum function $\mathrm{F}(R_p, I)$, where this functionis some probabilistic measure of distance between the returns of portfolio x and index I and can be interpreted as the measure of risk of the portfolio relative to index. AsGiavoronski et al. [18] mentioned this function can take different risk measures. Constraint (5) ensures that the maximum and minimum proportion level of chosen stocks in the portfolio satisfy proportion limit. Eq. (6) ensures that there are precisely K stocks in the tracking portfolio.Constraint (7) specified that all capital must be invested. Finally, constraints (8) and (9) denote the domain of the decision variables.

In this paper, we use VaR concept as risk measure in the objective function. Then, the model can be shown as follow:

$$\begin{cases} \min VaR_{1-\alpha} \\ st: \\ x^l z_i \le x_i \le x^u z_i \\ \sum_{i=1}^{n} z_i = K \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \ge 0 \\ z_i \in \{0,1\} \end{cases} \tag{10}$$

4. Fuzzy Index Tracking Model Based on Value-at-Risk

In this section, we propose fuzzy Index Tracking model based on Value-at-Risk. At first, some concepts about fuzzy variables and a survey of fuzzy distance is introduced. Then, fuzzy Value-at-Risk in some theorems is represented and by proving these theorems, fuzzy index tracking portfolio selection model based on Value-at-Risk is illustrated.

4.1. Some concepts about fuzzy variables

In this section, we have a brief review on credibility theory which was founded by Liu [19].

4.1.1. Credibility function

Let ξ be a fuzzy variable with membership function $\mu_\xi(x)$ and r be a real number. Then the credibility function of $\xi \ge r$ will be:

$$\mathrm{Cr}\{\xi \le r\} = \frac{1}{2}\left[\mathrm{Pos}\{\xi \le r\} + \mathrm{Nec}\{\xi \le r\}\right]. \tag{11}$$

where $\mathrm{Pos}\{.\}$ and $\mathrm{Nec}\{.\}$ are the possibility and necessity measures which are defined as follows:

$$\text{Pos}\{\xi \le r\} = \sup_{x \le r} \mu_\xi(x) \tag{12}$$

$$\text{Nec}\{\xi \le r\} = 1 - \sup_{x > r} \mu_\xi(x) \tag{13}$$

From Eq. (13) we have:

$$\text{Cr}\{\xi \le r\} = \frac{1}{2}\left[\sup_{x \le r} \mu_\xi(x) + 1 - \sup_{x > r} \mu_\xi(x)\right] \tag{14}$$

$$\text{Cr}\{\xi \ge r\} = \frac{1}{2}\left[\sup_{x \ge r} \mu_\xi(x) + 1 - \sup_{x < r} \mu_\xi(x)\right] \tag{15}$$

4.1.2. Graded mean integration representation

In 1998, Chen and Hsieh [20] proposed graded mean integration representation (GMIR) for representing generalized fuzzy number. They supposed L^{-1} and R^{-1} as inverse functions for a LR fuzzy number. They calculate GMIR of generalized fuzzy number A based on integral value of graded mean h-levels as follow:

$$P(A) = \int_0^w h\left(\frac{L^{-1}(h) + R^{-1}(h)}{2}\right) dh \Big/ \int_0^w h\ dh \tag{16}$$

where h is between 0 and w, and $0 < w \le 1$.

They have already found GMIR formula for a generalized trapezoidal fuzzy number $A = (a,b,c,d)$ and a generalized triangular fuzzy number $B = (a,b,c)$ as follows:

$$P(A) = \frac{a + 2b + 2c + d}{6} \tag{17}$$

$$P(B) = \frac{a + 4b + c}{6} \tag{18}$$

4.1.3. The fuzzy distance

Chen and Wang [1] represented a new fuzzy distance method by using the fuzzy absolute value of the difference of two trapezoidal fuzzy numbers. They supposed $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ as two trapezoidal fuzzy numbers with their GMIR $P(A)$ and $P(B)$. They assume:

$$s_i = \frac{\left(a_i - P(A) + b_i - P(B)\right)}{2}, \quad i = 1, 2, 3, 4 \tag{19}$$

Then they computed the fuzzy distance of A, B as follow:

$$C_i = d(A,B) = |A - B| = |P(A) - P(B)| + s_i, \quad i = 1, 2, 3, 4 \tag{20}$$

4.2. Fuzzy Value-at-Risk

Suppose $L = |R_p - I|$ represents the difference between portfolio and index returns. So, the VaR of L is the largest value by which the portfolio return can miss the index target in $1-\alpha$ fraction of cases(Giavoronski et al [18]) and can be written as follow:

$$\mathrm{VaR}_{1-\alpha} = \inf_{w} \left\{ w \,\middle|\, \mathrm{Cr}(L \le w) \ge \alpha \right\} \tag{21}$$

Using certain kinds of fuzzy variables, following theorems help us to solve model.

Theorem 1: Let $\xi_i = (a_i, b_i, c_i)$ be triangular fuzzy numbers for security returns with the membership function:

$$\mu_{\xi_i}(x) = \begin{cases} \dfrac{x_i - a_i}{b_i - a_i} & a_i \le x_i \le b_i \\ \dfrac{x_i - c_i}{b_i - c_i} & b_i \le x_i \le c_i \\ 0 & otherwise \end{cases} \tag{22}$$

So the $VaR_{1-\alpha}$ can be computed as follow:

$$VaR_{1-\alpha} = \begin{cases} (2\alpha - 1)a + (2 - 2\alpha)b & 0 < \alpha \le 0.5 \\ 2\alpha b + (1 - 2\alpha)c & 0.5 < \alpha \le 1 \end{cases} \tag{23}$$

Proof: in first step, we compute credibility function of fuzzy variable ξ which represents security returns. According to equations (12), (13) and (22), we have:

$$\mathrm{Pos}\{\xi \le r\} = \sup_{x \le r} \mu_{\xi}(x) = \begin{cases} \dfrac{x - a}{b - a} & x \le b \\ 1 & x > b \end{cases} \tag{24}$$

$$\mathrm{Nec}\{\xi \le r\} = 1 - \sup_{x > r} \mu_{\xi}(x) = \begin{cases} 0 & x \le b \\ 1 - \dfrac{x - c}{b - c} & x > b \end{cases} \tag{25}$$

So according to Eq. (14), the credibility function will be:

$$\mathrm{Cr}\{\xi \le r\} = \frac{1}{2}\left[\sup_{x \le r} \mu_{\xi}(x) + 1 - \sup_{x > r} \mu_{\xi}(x) \right] = \begin{cases} \dfrac{1}{2}\left(\dfrac{x - a}{b - a} \right) & x \le b \\ 1 - \dfrac{1}{2}\left(\dfrac{x - c}{b - c} \right) & x > b \end{cases} \tag{26}$$

Then,based on Eq. (21) we can calculate $VaR_{1-\alpha}$ for this variable as follow:

$$VaR_{1-\alpha} = \begin{cases} (2\alpha - 1)a + (2 - 2\alpha)b & 0 < \alpha \le 0.5 \\ 2\alpha b + (1 - 2\alpha)c & 0.5 < \alpha \le 1 \end{cases} \tag{27}$$

Theorem 2: Let $\xi_i = (a_i, b_i, c_i, d_i)$ be trapezoidal fuzzy numbers for security returns, and then similarly to theorem 1, $VaR_{1-\alpha}$ can be computed as follow:

$$VaR_{1-\alpha} = \begin{cases} (2\alpha - 1)a + (2 - 2\alpha)b & 0 < \alpha \le 0.5 \\ 2\alpha c + (1 - 2\alpha)d & 0.5 < \alpha \le 1 \end{cases} \tag{28}$$

Since $\xi_i = (a_i, b_i, c_i, d_i)$ represents trapezoidal fuzzy numbers for security returns and x_i shows proportion of security i in portfolio, according to Eq. (2), we can calculate portfolio's fuzzy return as follow:

$$R_p = \sum_{i=1}^{n} x_i \xi_i = \left(\sum_{i=1}^{n} x_i a_i , \sum_{i=1}^{n} x_i b_i , \sum_{i=1}^{n} x_i c_i , \sum_{i=1}^{n} x_i d_i \right) \tag{29}$$

Let $I = (a', b', c', d')$ be a trapezoidal fuzzy number for index return, then according to equations (17) and (19), we have:

$$\mathrm{P}(R_p) = \frac{\left(\sum_{i=1}^{n} x_i a_i + 2\sum_{i=1}^{n} x_i b_i + 2\sum_{i=1}^{n} x_i c_i + \sum_{i=1}^{n} x_i d_i \right)}{6} \tag{30}$$

$$\mathrm{P}(I) = \frac{\left(a' + 2b' + 2c' + d' \right)}{6} \tag{31}$$

$$s = \left(\begin{array}{l} \dfrac{\left(5\sum_{i=1}^{n} x_i a_i - 2\sum_{i=1}^{n} x_i b_i - 2\sum_{i=1}^{n} x_i c_i - \sum_{i=1}^{n} x_i d_i + 5a' - 2b' - 2c' - d' \right)}{12}, \\ \dfrac{\left(-\sum_{i=1}^{n} x_i a_i + 4\sum_{i=1}^{n} x_i b_i - 2\sum_{i=1}^{n} x_i c_i - \sum_{i=1}^{n} x_i d_i - a' + 4b' - 2c' - d' \right)}{12}, \\ \dfrac{\left(-\sum_{i=1}^{n} x_i a_i - 2\sum_{i=1}^{n} x_i b_i + 4\sum_{i=1}^{n} x_i c_i - \sum_{i=1}^{n} x_i d_i - a' - 2b' + 4c' - d' \right)}{12}, \\ \dfrac{\left(-\sum_{i=1}^{n} x_i a_i - 2\sum_{i=1}^{n} x_i b_i - 2\sum_{i=1}^{n} x_i c_i + 5\sum_{i=1}^{n} x_i d_i - a' - 2b' - 2c' + 5d' \right)}{12} \end{array} \right) \tag{32}$$

FromEq. (20) the loss function (L) which is the absolute difference between portfolio and index returns can be calculated as follow:

$$L=\left|R_p-I\right|=\left(\begin{array}{l}\left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(5\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i+5a'-2b'-2c'-d'\right)}{12},\\ \left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(-\sum_{i=1}^{n}x_ia_i+4\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i-a'+4b'-2c'-d'\right)}{12},\\ \left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(-\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i+4\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i-a'-2b'+4c'-d'\right)}{12},\\ \left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(-\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i+5\sum_{i=1}^{n}x_id_i-a'-2b'-2c'+5d'\right)}{12}\end{array}\right) \tag{33}$$

So, according to Eq. (28), we can compute $VaR_{1-\alpha}$ for L as below:

$$VaR_{1-\alpha}=\begin{cases}\begin{array}{l}(2\alpha-1)\left(\left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(5\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i+5a'-2b'-2c'-d'\right)}{12}\right)+\\ (2-2\alpha)\left(\left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(-\sum_{i=1}^{n}x_ia_i+4\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i-a'+4b'-2c'-d'\right)}{12}\right)\end{array} & 0<\alpha\le 0.5\\[2ex] \begin{array}{l}2\alpha\left(\left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(-\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i+4\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i-a'-2b'+4c'-d'\right)}{12}\right)+\\ (1-2\alpha)\left(\left|\dfrac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\dfrac{\left(a'+2b'+2c'+d'\right)}{6}\right|+\dfrac{\left(-\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i+5\sum_{i=1}^{n}x_id_i-a'-2b'-2c'+5d'\right)}{12}\right)\end{array} & 0.5<\alpha\le 1\end{cases} \tag{34}$$

The index tracking portfolio selection model for trapezoidal fuzzy numbers can be formulated as follows based on mentioned equations:

For $\alpha \le 0.5$ model (10) becomes:

$$\min\left(\begin{array}{l}(2\alpha-1)\left(\left|\frac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\frac{(d'+2b'+2c'+d')}{6}\right|+\frac{\left(5\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i+5d'-2b'-2c'-d'\right)}{12}\right)+\\ (2-2\alpha)\left(\left|\frac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\frac{(d'+2b'+2c'+d')}{6}\right|+\frac{\left(-\sum_{i=1}^{n}x_ia_i+4\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i-d'+4b'-2c'-d'\right)}{12}\right)\end{array}\right)$$

$$\begin{array}{l}st:\\ x^lz_i\le x_i\le x^uz_i\\ \sum_{i=1}^{n}z_i=K\\ \sum_{i=1}^{n}x_i=1\\ x_i\ge 0\\ z_i\in\{0,1\}\end{array}\qquad(35)$$

And for $\alpha > 0.5$ we have:

$$\min\left(\begin{array}{l}2\alpha\left(\left|\frac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\frac{(a'+2b'+2c'+d')}{6}\right|+\frac{\left(-\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i+4\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}x_id_i-a'-2b'+4c'-d'\right)}{12}\right)+\\ (1-2\alpha)\left(\left|\frac{\left(\sum_{i=1}^{n}x_ia_i+2\sum_{i=1}^{n}x_ib_i+2\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}x_id_i\right)}{6}-\frac{(a'+2b'+2c'+d')}{6}\right|+\frac{\left(-\sum_{i=1}^{n}x_ia_i-2\sum_{i=1}^{n}x_ib_i-2\sum_{i=1}^{n}x_ic_i+5\sum_{i=1}^{n}x_id_i-a'-2b'-2c'+5d'\right)}{12}\right)\end{array}\right)$$

$$\begin{array}{l}st:\\ x^lz_i\le x_i\le x^uz_i\\ \sum_{i=1}^{n}z_i=K\\ \sum_{i=1}^{n}x_i=1\\ x_i\ge 0\\ z_i\in\{0,1\}\end{array}\qquad(36)$$

5. Numerical example

In order to illustrate the proposed method, we consider the following numerical examplebased on Tehran Stock Exchange (TSE) market data. For this mean, we consider 80 stocks datafrom March 26, 2011 to June 20, 2012which hadthe most transaction day.In order to construct an index tracking portfolio with fuzzy returns based on VaR, we apply model (36). For this purpose, we used the data set from March 26, 2011 to December 21, 2011 as learning data set. We consider these stocks returns as trapezoidal fuzzy number $\xi_i=(a_i,b_i,c_i,d_i)$ where:

a_i is stock i^{th} return at 5^{th} percentile, b_i is stock i^{th} return at 40^{th} percentile, c_i is stock i^{th} return at 60^{th} percentile and d_i is stock i^{th} return at 95^{th} percentile.

According to the above explanation, stocks fuzzy returns were determined as follows:

Table 1. Stocks Fuzzy Returns

Security No.	**Fuzzy Return**	**Security No.**	**Fuzzy Return**
1	(-0.0143,-0.0012,-0.0003,0.0171)	41	(-0.0353,-0.0024,0.0004,0.0341)
2	(-0.0404,-0.0068,0.0018,0.0391)	42	(-0.0369,-0.007,-0.0007,0.0336)
3	(-0.0378,-0.0063,0.0042,0.0389)	43	(-0.0393,-0.0076,0.0022,0.0389)
4	(-0.0179,-0.0012,0,0.0185)	44	(-0.0222,-0.0018,-0.0003,0.0361)
5	(-0.0231,-0.0021,0,0.0204)	45	(-0.0366,0,0.0023,0.0384)
6	(-0.02104,-0.0013,0,0.02846)	46	(-0.0203,-0.0016,0.0009,0.0381)
7	(-0.0321,-0.0048,0.0012,0.0352)	47	(-0.0274,-0.0005,0.0001,0.0375)
8	(-0.0267,-0.0019,0.0012,0.0328)	48	(-0.0388,-0.0016,0.0016,0.0384)
9	(-0.04,-0.0085,0.0038,0.0389)	49	(-0.0103,-0.001,0,0.0168)
10	(-0.0367,-0.0062,0.0066,0.0381)	50	(-0.0135,-0.0014,0,0.0195)
11	(-0.0393,-0.0103,0.0073,0.0388)	51	(-0.0148,-0.0009,0,0.0318)
12	(-0.0398,-0.0058,0.0077,0.0389)	52	(-0.0293,-0.0016,0,0.0388)
13	(-0.0327,-0.0034,0,0.0348)	53	(-0.0378,-0.0079,0,0.0379)
14	(-0.0385,-0.0048,0.0008,0.039)	54	()-0.0285,-0.0008,0.0006,0.0389
15	(-0.0363,-0.0047,0.0004,0.037)	55	(-0.029,-0.0021,-0.0002,0.039)
16	(-0.0236,-0.001,0,0.0122)	56	(-0.0319,-0.0039,-0.0005,0.0372)
17	(-0.0321,-0.003,-0.0003,0.0289)	57	(-0.0322,-0.0021,0,0.0377)
18	(-0.0388,-0.0034,0.003,0.0392)	58	(-0.034,-0.003,0,0.0357)
19	(-0.0391,-0.0071,0.0148,0.0392)	59	(-0.0247,-0.0024,-0.0004,0.0367)
20	(-0.0269,-0.002,0.0002,0.0363)	60	(-0.0379,-0.0036,0,0.0377)
21	(-0.0367,-0.001,0.0023,0.0386)	61	(-0.0343,-0.0048,0.0033,0.0372)
22	(-0.034,-0.0011,0.0018,0.0388)	62	(-0.0348,-0.0022,0.0044,0.0368)
23	(-0.0386,-0.0036,0.008,0.0388)	63	(-0.0297,-0.0027,0.0004,0.0282)
24	(-0.0385,-0.0055,0.0052,0.391)	64	(-0.0092,-0.0008,0,0.0159)
25	(-0.0362,-0.0096,0.0009,0.0374)	65	(-0.0332,-0.0035,0.0004,0.0345)
26	(-0.0361,-0.0066,0.0011,0.0385)	66	(-0.0362,-0.0022,0.0017,0.0375)
27	(-0.0371,-0.0075,0.0015,0.0375)	67	(-0.0336,-0.009,0.003,0.0369)
28	(-0.0367,-0.0061,0.0036,0.0371)	68	(-0.0242,-0.0025,-0.0006,0.0356)
29	(-0.0392,-0.0054,0.0021,0.0373)	69	(-0.0178,-0.0013,0,0.0006)
30	(-0.0261,-0.00025,0,0.0245)	70	(-0.0367,-0.0051,-0.0001,0.0369)
31	(-0.0369,-0.0057,0.0009,0.0377)	71	(-0.0374,-0.003,0.0023,0.0377)
32	(-0.0297,-0.0019,0.0008,0.0271)	72	(-0.0237,-0.0018,0,0.0325)
33	(-0.0328,-0.0018,0.0003,0.0376)	73	(-0.033,-0.0051,0.0025,0.0339)
34	(-0.0388,-0.0106,0.0007,0.0375)	74	(-0.0335,-0.0022,0.008,0.0367)
35	(-0.0349,-0.0034,0.0003,0.033)	75	(-0.0246,-0.0013,0.0003,0.0311)
36	(-0.0356,-0.0026,0.0008,0.0376)	76	(-0.0117,-0.0016,0,0.02)
37	(-0.0313,-0.0041,0,0.0349)	77	(-0.0389,-0.0058,0.074,0.0389)
38	(-0.0353,-0.0043,0.0003,0.0371)	78	(-0.025,-0.0023,-0.0009,0.0305)
39	(-0.0376,-0.0101,0.0038,0.0375)	79	(-0.0343,-0.0021,0,0.0377)
40	(-0.0388,-0.0078,0.0056,0.0388)	80	(-0.0246,-0.0029,0,0.0302)

In order to test the portfolio's efficiency, we used the test data sets from December 22, 2011 to June 20, 2012 and solve this model with different K , x^l and x^u . Results are illustrated in table 2.These different index tracking portfolios' return paths are illustrated in figures 2, 3, 4, 5, 6, 7 and 8.

Limitation		Securities Proportion															
		X1	X4	X6	X19	X44	X46	X49	X50	X51	X60	X65	X69	X73	X77	X80	RMSE
K=5	x^l =0.05 x^u =0.50						0.05	0.05		0.50		0.35			0.05		0.007988
K=5	x^l =0.10 x^u =0.50						0.10	0.10		0.50		0.20			0.10		0.008107
K=10	x^l =0.01 x^u =0.50	0.01			0.01	0.01	0.01	0.01	0.01	0.50		0.42			0.01	0.01	0.008073
K=10	x^l =0.05 x^u =0.50	0.05			0.05	0.05	0.05	0.05	0.05	0.50		0.10			0.05	0.05	0.007173
K=15	x^l =0.01 x^u =0.50	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.50	0.01	0.37	0.01	0.01	0.01	0.01	0.007825
K=15	x^l =0.03 x^u =0.50	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.50	0.03	0.11	0.03	0.03	0.03	0.03	0.006989
K=15	x^l =0.05 x^u =0.50	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.30	0.05	0.05	0.05	0.05	0.05	0.05	0.005285

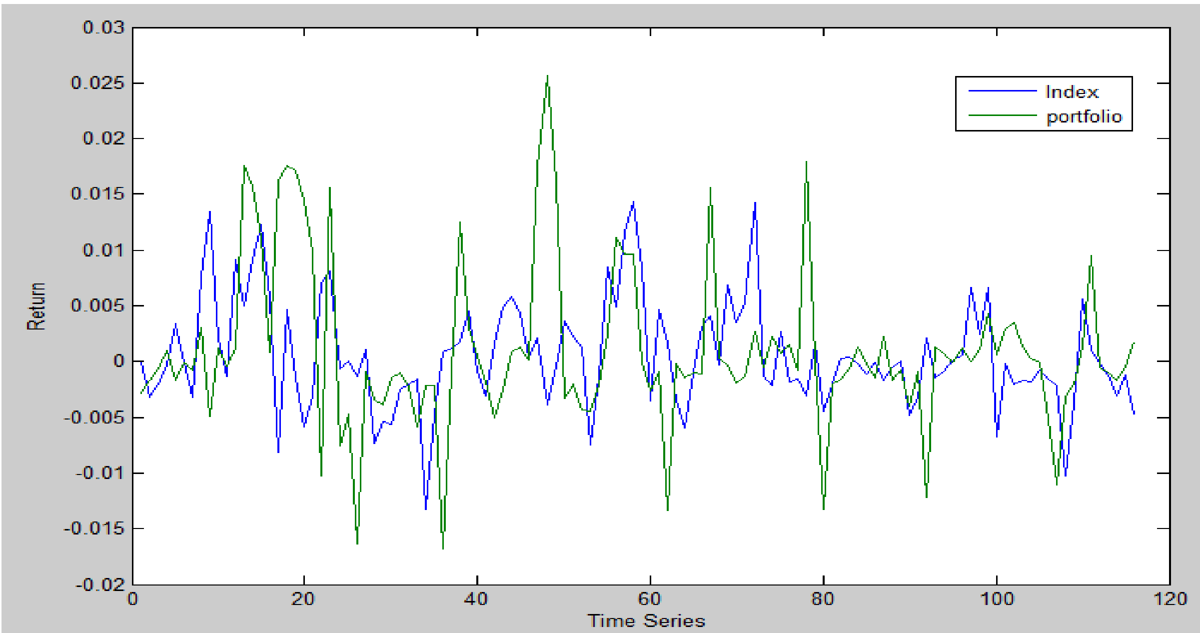

Figure 1. Index Tracking Portfolio with $k = 5$, $x^l = 0.05$, $x^u = 0.5$

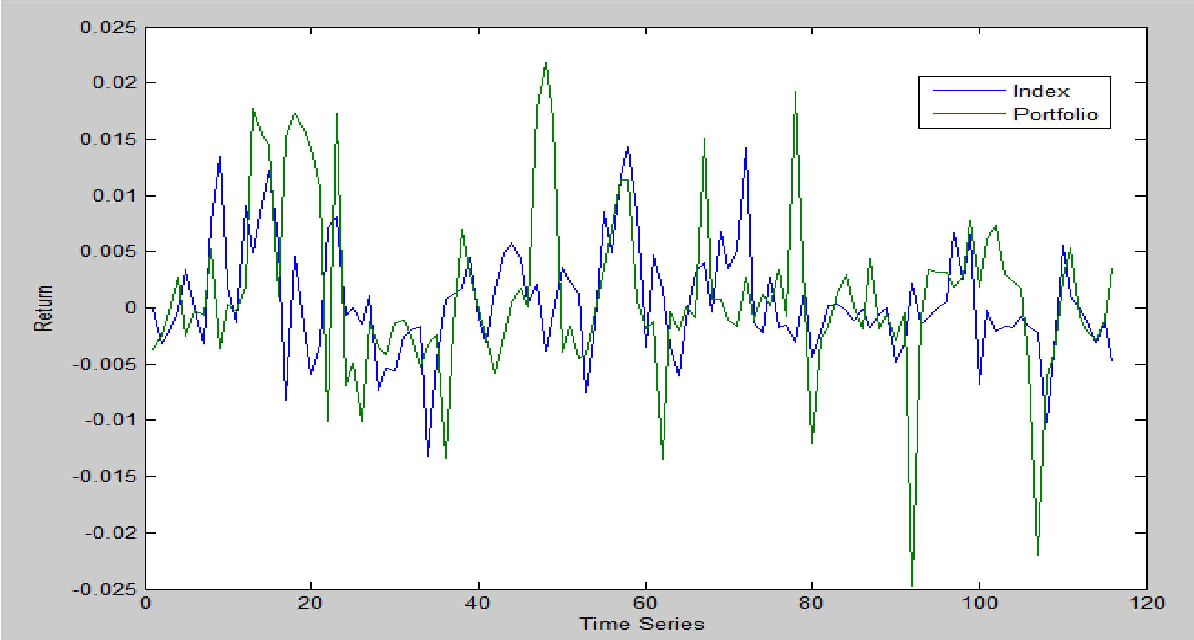

Figure 2. Index Tracking Portfolio with $k = 5$, $x^l = 0.1$, $x^u = 0.5$

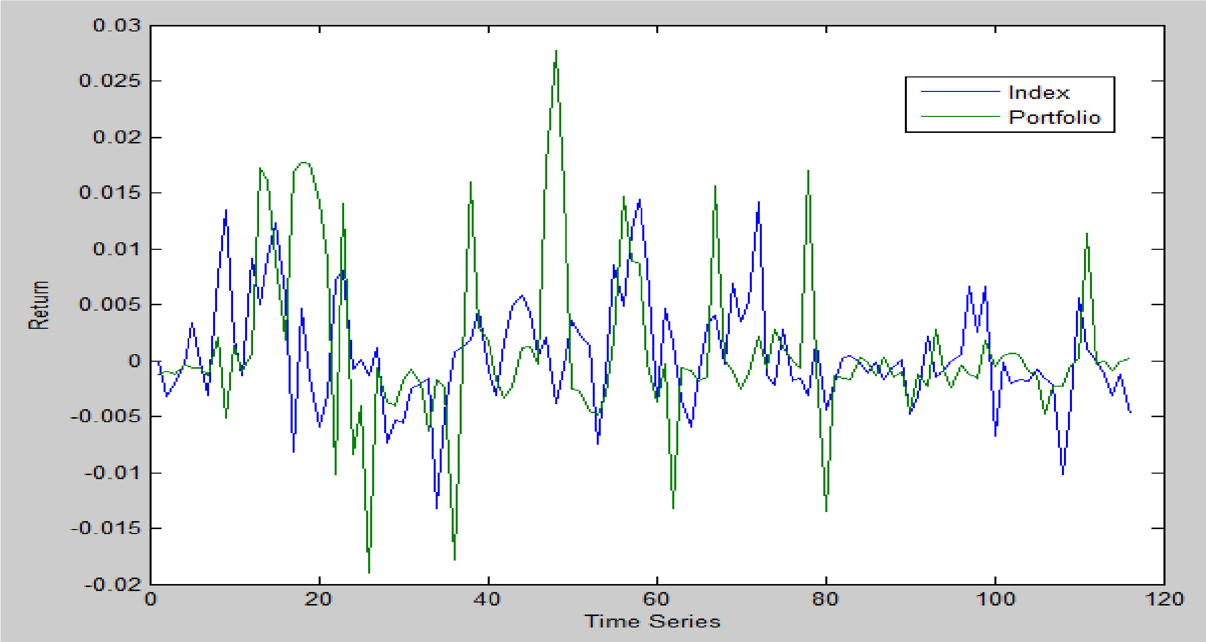

Figure 3. Index Tracking Portfolio with $k = 10$, $x^l = 0.01$, $x^u = 0.5$

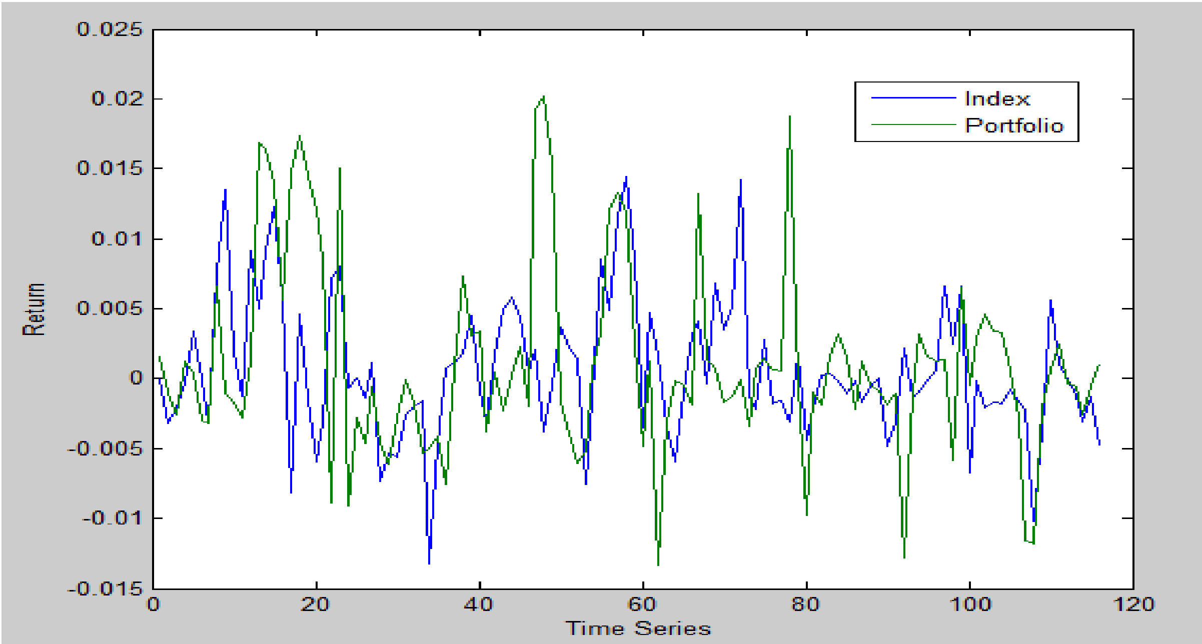

Figure 4. Index Tracking Portfolio with $k = 10$, $x^l = 0.05$, $x^u = 0.5$

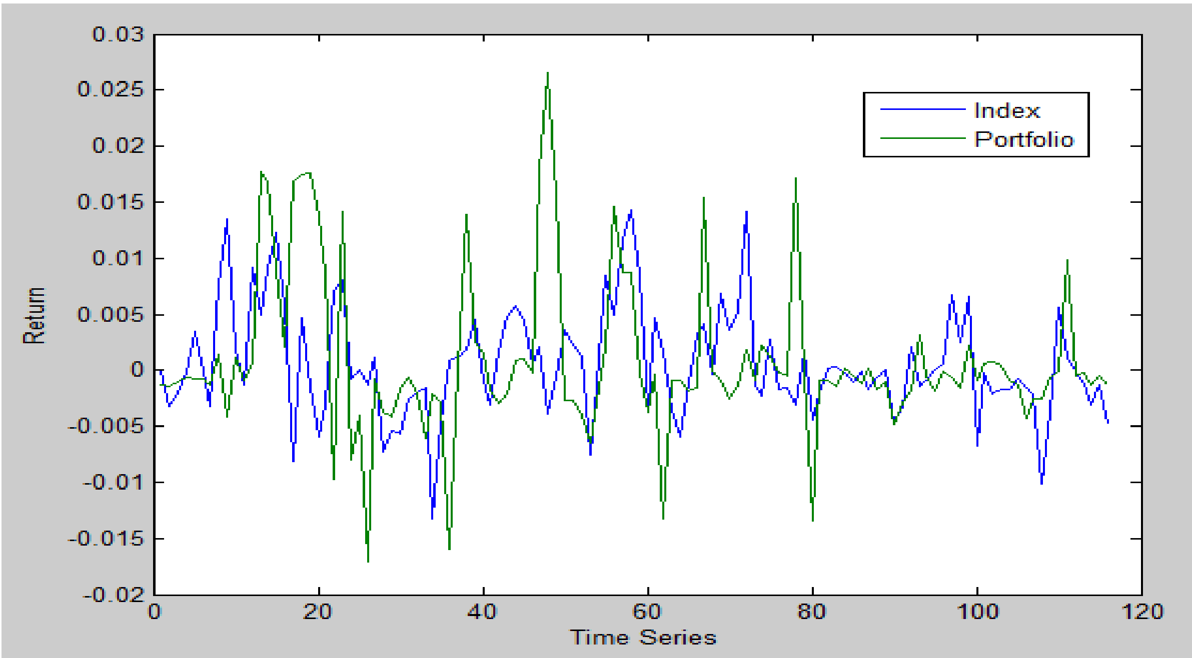

Figure 5. Index Tracking Portfolio with $k = 15$, $x^l = 0.01$, $x^u = 0.5$

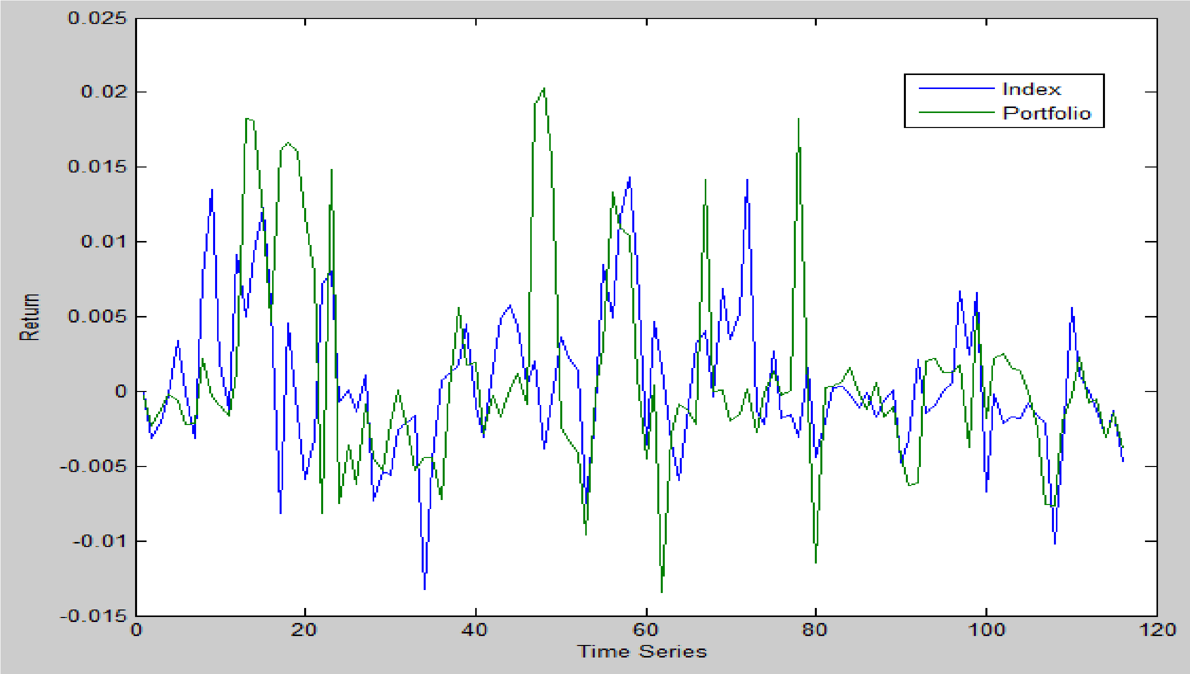

Figure 6. Index Tracking Portfolio with $k = 15$, $x^l = 0.03$, $x^u = 0.5$

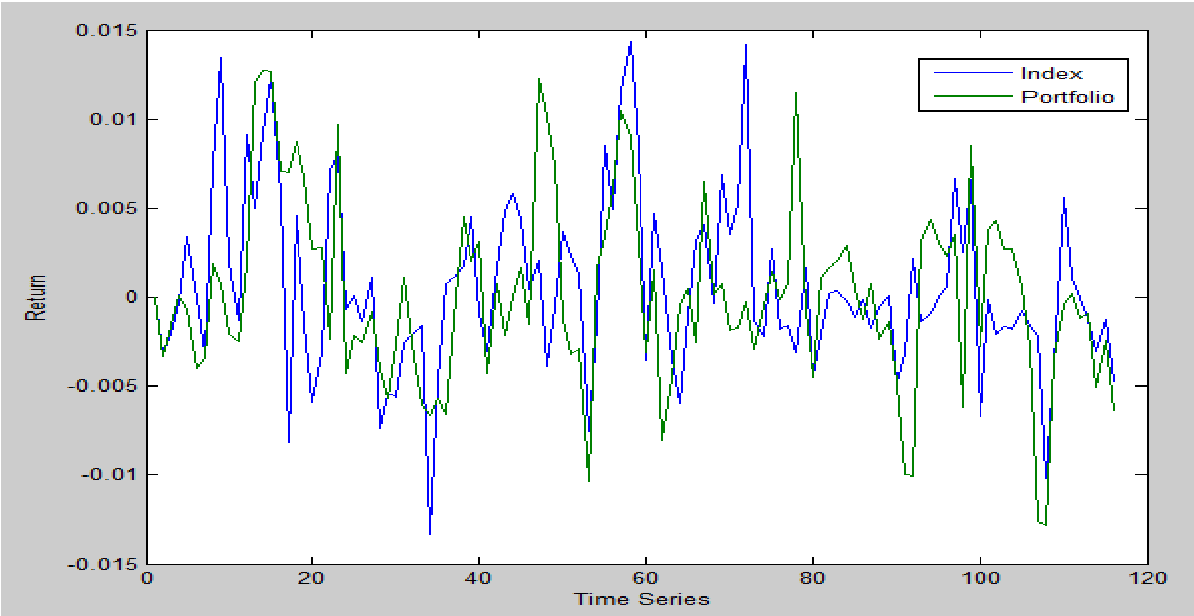

Figure 7. Index Tracking Portfolio with $k = 15$, $x^l = 0.05$, $x^u = 0.5$

These result shows that the index tracking portfolio with $k = 15$, $x^l = 0.05$, $x^u = 0.5$ has the lowest RMSE criteria and shows better fitness to the Tehran Price Index (TEPIX).

6. Conclusion

Most recent studies determined security returns in conventional models by the precise historical data. However, such precise data are not always available. Therefore, to handle such imprecise uncertainty,considering security returns as variables with imprecise distributions, i.e., fuzzy variables are recommended.In order to make a portfolio which tracks a specific index performance, researchers have studied and experimented various risk-measure methods. Nevertheless, index tracking portfolio selection model with VaR have not been yet established.In this research, we applied index tracking portfolio selection model with fuzzy VaR.To solve the model by using certain kinds of fuzzy variables, we have proved some theorems. Finally the model was tested by TSE market data (TEPIX). Results show that the model has low RMSE criteria and can be applicable to various investment problems when the future returns are uncertain.

References

[1] Wang, S., Watada, J. and Pedrycz, W. (2009), Value-at-risk-based two-stage fuzzy facility location problems, *IEEE Transactions on Industrial Informatics*, Vol. 5, No. 4, pp. 465-482.

[2] Holdges, S.D. (1976), Problems in the application of portfolio selection models, *Omega*, Vol. 4, No. 6, pp. 699-709.

[3] Roll, R. (1992), A mean-variance analysis of tracking error, *The Journal of Portfolio Management*, Vol. 18, No. 4, pp. 13-22.

[4] Rohweder, H.C. (1998), Implementing stock selection ideas: Does tracking error optimization do any good?, *The Journal of PortfolioManagement*, Vol. 24, No. 3, pp. 49-59.

[5] Wang, M.Y. (1999), Multiple-benchmark and multiple-portfolio optimization, *Financial Analysts Journal*, Vol. 55, No. 1, pp. 63-72.

[6] Beasley, J.E., Meade, N. and Chang, T.J. (2003), An evolutionary heuristic for the index tracking problem, *European Journal of Operational Research*, Vol. 148, No. 3, pp. 621-643.

[7] Rudd, A. (1980), Optimal selection of passive portfolios, *Financial Management*, Vol. 9, No. 1, pp. 57-66.

[8] Corielli, F. and Marcellino, M. (2005), Factor based index tracking, *Journal of Banking and Finance*, Vol. 30, No. 8, pp. 2215-2233.

[9] Okay, N. and Akman, U. (2003), Index tracking with constraint aggregation, *Applied Economics Letters*, Vol. 10, No. 14, pp. 913-916.

[10] Oh, J.K., Kim, T.Y. and Min, S. (2005), Using Genetic Algorithm to Support Portfolio Optimization for Index Fund Management, *Expert System with Applications*, Vol. 28, No. 2, pp. 371-379.

[11] Ferino, A., Gallagher, D.R., and Oetomo, T.N. (2005), The index tracking strategies of passive and enhanced index equity funds, *Australian Journal of Management*, Vol. 30, No. 1, pp. 23-55.

[12] Krink, T., Mitnnik, S. and Paterlini, S. (2007), Differential evolution and combinatorial search for constrained index tracking, *Annals of Operations Research*, Vol. 172, No.1, pp. 153-176 .

[13] Torrubiano, R.R. and Alberto, S. (2009), A hybrid optimization approach to index tracking, *Annals of Operational Research*, Vol. 166, No. 1, pp. 57-71.

[14] Wang, B., Wang, S. and Watada, J. (2011), Fuzzy Portfolio Selection Models with Value-at-Risk, *IEEE Transactionson Fuzzy Systems*, Vol. 19, No. 4, pp. 758-769.

[15] Watada, J. (1997), Fuzzy portfolio selection and its application to decision making, *Tatra Mountains Math Publication*, Vol. 13, No. 4, pp. 219-248.

[16] Gazanfar-Ahari, S., Ghaffari-Nasab, N. Makui, A. and Ghodsypour, S.H. (2011), A portfolio selection using fuzzy analytic hierarchy process: A case study of Iranian pharmaceutical industry, *International Journal of Industrial Engineering Computations*, Vol. 2, No. 2, pp. 225-236.

[17] Alimi, A., Zandieh, M. and Amiri, M. (2012), Multi-objective portfolio optimization of mutual funds under downside risk measure using fuzzy theory, *International Journal of Industrial Engineering Computations*, Vol. 3, No. 5, pp. 859-872.

[18] Gaivoronski, A.A., Krylov, S. and Wijst, N. (2005), Optimal Portfolio Selection and Dynamic Benchmark Tracking, *European Journal of Operational Research*, Vol. 163, No. 1, pp. 1115-131.

[19] Liu, B. and Liu, Y.K. (2002), Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy System*, Vol. 10, No. 4, pp. 445-450.

[20] Chen, S.H. and Hsieh, C.H. (1999), Graded mean integration representation of generalized fuzzy number, *Journal of the Chinese Fuzzy System Association*, Vol. 5, No. 2, pp. 1-7.