A New Method for Ranking Efficient DMUs Based on TOPSIS and Virtual DMUs

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ARTICLE INFO

Article history:
Received:
January 31, 2012
Revised:
May 15, 2012
Accepted:
June 10, 2012

ABSTRACT

In this paper, a method for ranking efficient DMUs based on TOPSIS has been proposed. The difference between the distance of the center of gravity of all efficient DMUs to the ideal point and the anti-ideal point after and before deleting efficient DMUs one by one is the criteria of ranking efficient DMUs. In this paper, the proposed method is compared with AP (input oriented), MAJ (input oriented), AP (output oriented), MAJ (output oriented) models and norm method. This comparison shows that the proposed method is better than the above-mentioned models. The proposed method is also always feasible and simpler in comparison with other methods.

Keywords:
Data envelopment analysis; Decision making unit (DMU); Ranking models; TOPSIS.

1. Introduction

Data Envelopment Analysis (DEA) is a methodology for estimating the efficiency of decision making units (DMUs) with multiple inputs and outputs. For the first time, Charnes et al. [1], introduced a method for determining the efficiency of decision making units. In 1993, Anderson and Peterson [2] proposed the super-efficiency method for ranking efficient DMUs (specifying the best DMU among all efficient DMUs). Mehrabian et al. (MAJ) [3] have modified the AP model. Jahanshaloo et al. [4] presented the method for ranking efficient DMUs by $l_1$ norm. Also Zhu presented a ranking method, named PCA [5]. The TOPSIS (technique for order performance (preference) by similarity to ideal solution) is a method for ranking DMUs which was presented by Hwang & Yoon (1981) [6]. TOPSIS ranks DMUs by calculating the distances of DMUs to the ideal point and the anti-ideal point, in which each DMU with maximum distance to the ideal point and minimum distance to anti-ideal point receives rank 1 and, every DMU with minimum distance to the ideal point and maximum distance to anti-ideal point receives rank n. This paper is organized into following sections: Insection 2, we provide an overview of data envelopment analysis and ranking models. In section 3, we introduce an approach for ranking efficient DMUs, then a numerical example is given in section 4, and finally in section 5 conclusions are drawn.

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2. An overview of Data Envelopment Analysis, ranking models and TOPSIS

2.1. CCR model

Data Envelopment Analysis is a method which is applied for obtaining efficiency of DMUs. Let us have \( n \) DMUs which have \( m \) inputs and \( s \) outputs, suppose \( \{x_{1j}, x_{2j}, ..., x_{mj}\} \) and \( \{y_{1j}, y_{2j}, ..., y_{sj}\} \) represent input and output vectors of DMU\(_j\), respectively. For evaluation of the efficiency of DMU\(_a\), CCR model is given below:

\[
\text{MAX} \sum_{r=1}^{s} u_r y_{ro} \\
\text{s.t.} \sum_{i=1}^{m} v_i x_{io} = 1 \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad , j = 1, ..., n \\
u_r \geq 0 \quad , r = 1, ..., s \quad , \quad v_i \geq 0 \quad , i = 1, ..., m
\]

where, \( v_i \) and \( u_r \) represent the weights of \( i^{th} \) input and \( r^{th} \) output, respectively.

2.2. AP model

Anderson and Peterson [2] proposed Super Efficiency model. This model ranks DMUs by deleting DMU under evaluation from Production Possible Set (PPS) and offers the DEA model for other DMUs.

\[
\text{Max} \sum_{r=1}^{s} u_r y_{ro} \\
\text{s.t.} \sum_{i=1}^{m} v_i x_{io} = 1 \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad , j = 1, ..., n , j \neq o \\
u_r \geq 0 \quad , r = 1, ..., s \quad , \quad v_i \geq 0 \quad , i = 1, ..., m
\]

2.3. MAJ model

To eliminate the major problems of AP model, Mehrabian et al [3], proposed another model for ranking efficient DMUs. Their proposed model follows here:

\[
\text{Min} 1 + w
\]
\[ s.t. \quad \sum_{j=1, j \neq o}^{n} \mu_j x_{ij} \leq x_{io} + w, \quad i = 1, \ldots, m \]

\[
\sum_{j=1, j \neq o}^{n} \mu_j y_{rj} \geq y_{ro} \quad , r = 1, \ldots, s
\]  

(3)

\[
\mu_j \geq 0 \quad , \quad j = 1, \ldots, n \quad , j \neq o
\]

### 2.4. Rank by \(l_1\) norm

Jahanshaloo et al [4], used \(l_1\) norm for ranking extremely efficient DMUs. This model comes next:

\[
\text{Min} \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + \alpha
\]

\[s.t. \quad \sum_{j=1, j \neq o}^{n} \mu_j x_{ij} \leq x_{i} + w, \quad i = 1, \ldots, m \]

\[
\sum_{j=1, j \neq o}^{n} \mu_j y_{rj} \geq y_{r} \quad , r = 1, \ldots, s
\]  

(4)

\[
x_i \geq x_{io} \quad , \quad i = 1, \ldots, m
\]

\[
y_r \geq y_{ro} \quad , \quad r = 1, \ldots, s
\]

\[
\mu_j \geq 0 \quad , \quad j = 1, \ldots, n \quad , j \neq o
\]

where, \( \alpha = \sum_{r=1}^{s} y_{ro} - \sum_{i=1}^{m} x_{io} \) is a constant value. For a better understanding of the MAJ model, refer to Jahanshaloo et al (2004).

### 2.5. Information on PCA

The ranking method proposed by Zhu [5] is presented below:

\[
d_{ri}^{j} = \frac{y_{rj}}{x_{ij}} \quad , \quad i = 1, \ldots, m \quad , \quad r = 1, \ldots, s
\]  

(5)

The higher \( d_{ri}^{j} \), shows a better performance of \( DMU_j \) in terms of the \( r^{th} \) output and \( t^{th} \) input compared with other DMUs.
2.6. TOPSIS

Ideal point is called $DMU^+$ and anti-ideal point is called $DMU^-$. Suppose $\{x_1^n, x_2^+, ..., x_m^n\}$ and $\{y_1^+, y_2^+, ..., y_s^+\}$ represent input and output vectors of $DMU^+$, and suppose $\{x_1^-, x_2^-, ..., x_m^-\}$ and $\{y_1^-, y_2^-, ..., y_s^-\}$ represent input and output vectors of $DMU^-$, respectively. Consider the following relations:

\[
\begin{align*}
  x_i^+ &= \min_{j=1}^n x_{ij}, \quad i = 1, ..., m \\
  y_r^+ &= \max_{j=1}^m x_{rj}, \quad r = 1, ..., s \\
  x_i^- &= \max_{j=1}^n x_{ij}, \quad i = 1, ..., m \\
  y_r^- &= \min_{j=1}^m x_{rj}, \quad r = 1, ..., s
\end{align*}
\]

(6)

$DMU_j^+$'s distance to the ideal point is calculated as follows:

\[
D_j^+ = \sqrt{\sum_{i=1}^m (x_{ij} - x_i^+)^2 - \sum_{r=1}^s (y_{rj} - y_r^+)^2}
\]

(7)

$DMU_j^-$'s distance to the anti-ideal point is calculated as follows:

\[
D_j^- = \sqrt{\sum_{i=1}^m (x_{ij} - x_i^-)^2 - \sum_{r=1}^s (y_{rj} - y_r^-)^2}
\]

(8)

Now, $DMU_j^-$'s general distance to the ideal point and the anti-ideal point has been provided here:

\[
D_j = D_j^+ / D_j^- + D_j^-
\]

(9)

The larger value of $D_j$, represents better performance of $DMU_j$ compared with other $DMUs$.

For more information about the TOPSIS method, refer to [6].

3. A New Method for ranking Efficient DMUs Based on TOPSIS and Virtual DMUs

In this new method, first the ideal point and the anti-ideal point are obtained using relation (6), all the efficient DMUs are identified, and then obtained results are put in $E = \{1, \ldots, e\}$. Continuing with all the efficient DMUs, we produced a virtual DMU. The acquired inputs and outputs of the virtual DMU are the average of the corresponding inputs and outputs of all the efficient DMUs, respectively. And it is called $DMU_M$ ($DMU_M$ is the center of gravity of all the efficient DMUs). After that, general distance of $DMU_M$ to the ideal point and the anti-ideal point are obtained through applying relation (9), this distance is called $D_M^+$. In the next
Following the procedure, the true difference between $DMU_{q,i}$ is deleted from the efficient frontier and the new virtual DMU is created which is the average of the corresponding inputs and outputs of all the efficient DMUs, respectively, except $DMU_{q, i}$. This is called $DMU_{q, i}$. After that, general distance of $DMU_{q, i}$ to the ideal point and the anti-ideal point is obtained using relation (9), and it is called $D_{q, i}$. This method will be repeated for every $DMU_{q, i}, q \in E$. The distance $d_{q, i}$ is the criteria for ranking efficient DMUs, the larger is $d_{q, i}$ the better ranking $DMU_{q, i}$ will have.

**Algorithm**

**Step1.** The ideal point and the anti-ideal point are obtained using relation (6).

**Step2.** We identify all the efficient DMUs and put them in $E = \{1, \ldots, e\}$.

**Step3.** Using all the efficient DMUs, we produce a virtual DMU, the inputs and outputs of the virtual DMU are the average of the corresponding inputs and outputs of all the efficient DMUs, respectively. It is called $DMU_M$ ($DMU_M$ is the center of gravity of all the efficient DMUs).

**Step4.** General distance of $DMU_M$ to the ideal point and the anti-ideal point are obtained applying relation (9), which is called $D_M$.

**Step5.** $DMU_{q, i}, q \in E$ is deleted from the efficient frontier and the new virtual DMU is created from the average of the corresponding inputs and outputs of all the efficient DMUs, with the exception of $DMU_{q, i}$, which is called $DMU_{q, i}'$.

**Step6.** The general distance $DMU_{q, i}'$ to the ideal point and the anti-ideal point is obtained through relation (9), and it is called $D_{q, i}'$.

**Step7.** Difference between $D_M$ and $D_{q, i}'$ will be computed and named as $d_{q, i}'$.

**Step8.** This method will be repeated for every $DMU_{q, i}, q \in E$.

**Step9.** $d_{q, i}'$ is the criteria for ranking efficient DMUs, the larger is $d_{q, i}'$, the better ranking $DMU_{q, i}$ will have.
4. Numerical Example

To illustrate the proposed ranking model, we consider an example with 28 DMUs which have 3 inputs and 3 outputs [4]. Consider Table 1:

<table>
<thead>
<tr>
<th>DMU</th>
<th>Input 1</th>
<th>Input 2</th>
<th>Input 3</th>
<th>Output 1</th>
<th>Output 2</th>
<th>Output 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>483.01</td>
<td>1,397,736</td>
<td>616,961</td>
<td>6,785,798</td>
<td>1,594,957</td>
<td>1,088,699</td>
</tr>
<tr>
<td>2</td>
<td>371.95</td>
<td>855,509</td>
<td>385,453</td>
<td>2,505,984</td>
<td>545,140</td>
<td>835,745</td>
</tr>
<tr>
<td>3</td>
<td>268.23</td>
<td>685,584</td>
<td>341,941</td>
<td>2,292,025</td>
<td>406,947</td>
<td>473,600</td>
</tr>
<tr>
<td>4</td>
<td>202.02</td>
<td>452,713</td>
<td>117,429</td>
<td>1,158,016</td>
<td>135,939</td>
<td>336,165</td>
</tr>
<tr>
<td>5</td>
<td>197.93</td>
<td>471,650</td>
<td>112,634</td>
<td>1,244,124</td>
<td>204,909</td>
<td>317,709</td>
</tr>
<tr>
<td>6</td>
<td>178.96</td>
<td>423,124</td>
<td>189,743</td>
<td>1,187,130</td>
<td>190,178</td>
<td>605,037</td>
</tr>
<tr>
<td>7</td>
<td>148.04</td>
<td>367,012</td>
<td>97,004</td>
<td>658,910</td>
<td>86,514</td>
<td>239,760</td>
</tr>
<tr>
<td>8</td>
<td>189.93</td>
<td>408,311</td>
<td>111,904</td>
<td>993,238</td>
<td>1,411,954</td>
<td>353,896</td>
</tr>
<tr>
<td>9</td>
<td>23.33</td>
<td>245,542</td>
<td>91,861</td>
<td>854,188</td>
<td>135,327</td>
<td>239,360</td>
</tr>
<tr>
<td>10</td>
<td>116.91</td>
<td>305,316</td>
<td>91,710</td>
<td>606,743</td>
<td>78,357</td>
<td>208,188</td>
</tr>
<tr>
<td>11</td>
<td>129.62</td>
<td>295,812</td>
<td>92,409</td>
<td>736,545</td>
<td>114,365</td>
<td>298,112</td>
</tr>
<tr>
<td>12</td>
<td>106.26</td>
<td>198,703</td>
<td>53,499</td>
<td>454,684</td>
<td>67,154</td>
<td>233,733</td>
</tr>
<tr>
<td>13</td>
<td>89.70</td>
<td>210,891</td>
<td>95,642</td>
<td>494,196</td>
<td>78,992</td>
<td>118,553</td>
</tr>
<tr>
<td>14</td>
<td>109.26</td>
<td>282,209</td>
<td>84,202</td>
<td>842,854</td>
<td>149,186</td>
<td>243,361</td>
</tr>
<tr>
<td>15</td>
<td>85.50</td>
<td>184,992</td>
<td>49,357</td>
<td>776,285</td>
<td>116,974</td>
<td>234,875</td>
</tr>
<tr>
<td>16</td>
<td>72.17</td>
<td>222,327</td>
<td>73,907</td>
<td>490,998</td>
<td>117,854</td>
<td>118,924</td>
</tr>
<tr>
<td>17</td>
<td>76.18</td>
<td>161,159</td>
<td>47,977</td>
<td>482,448</td>
<td>67,857</td>
<td>158,250</td>
</tr>
<tr>
<td>18</td>
<td>73.21</td>
<td>144,163</td>
<td>43,312</td>
<td>515,237</td>
<td>114,883</td>
<td>101,231</td>
</tr>
<tr>
<td>19</td>
<td>86.72</td>
<td>190,043</td>
<td>55,326</td>
<td>625,514</td>
<td>173,099</td>
<td>130,423</td>
</tr>
<tr>
<td>20</td>
<td>69.09</td>
<td>158,439</td>
<td>66,460</td>
<td>382,880</td>
<td>74,126</td>
<td>123,968</td>
</tr>
<tr>
<td>21</td>
<td>77.69</td>
<td>135,046</td>
<td>46,198</td>
<td>867,467</td>
<td>65,229</td>
<td>262,876</td>
</tr>
<tr>
<td>22</td>
<td>97.42</td>
<td>206,926</td>
<td>66,120</td>
<td>830,142</td>
<td>128,279</td>
<td>242,773</td>
</tr>
<tr>
<td>23</td>
<td>54.96</td>
<td>79,563</td>
<td>43,192</td>
<td>521,684</td>
<td>37,245</td>
<td>184,055</td>
</tr>
<tr>
<td>24</td>
<td>67.00</td>
<td>144,092</td>
<td>43,350</td>
<td>869,973</td>
<td>86,859</td>
<td>194,416</td>
</tr>
<tr>
<td>25</td>
<td>46.30</td>
<td>100,431</td>
<td>31,428</td>
<td>604,715</td>
<td>55,989</td>
<td>127,586</td>
</tr>
<tr>
<td>26</td>
<td>65.12</td>
<td>96,873</td>
<td>28,112</td>
<td>601,299</td>
<td>37,088</td>
<td>224,855</td>
</tr>
<tr>
<td>27</td>
<td>20.09</td>
<td>50,717</td>
<td>54,650</td>
<td>145,792</td>
<td>11,816</td>
<td>24,442</td>
</tr>
<tr>
<td>28</td>
<td>69.81</td>
<td>117,790</td>
<td>30,976</td>
<td>319,218</td>
<td>31,726</td>
<td>169,051</td>
</tr>
</tbody>
</table>

The ideal point and the anti-ideal point obtained by relation (6).

<table>
<thead>
<tr>
<th></th>
<th>Input 1</th>
<th>Input 2</th>
<th>Input 3</th>
<th>Output 1</th>
<th>Output 2</th>
<th>Output 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal</td>
<td>20.09</td>
<td>50717</td>
<td>28112</td>
<td>6785798</td>
<td>1594957</td>
<td>1088699</td>
</tr>
<tr>
<td>Anti-ideal</td>
<td>483.01</td>
<td>1397736</td>
<td>616961</td>
<td>145792</td>
<td>11816</td>
<td>24442</td>
</tr>
</tbody>
</table>
Using CCR model, unit 1, unit 2, unit 6, unit 8, unit 21, unit 23, unit 24, unit 25, unit 26 and unit 27 have become efficient. The efficient DMUs are ranked by the new method. The results of ranking are shown in Table 3.

<table>
<thead>
<tr>
<th>Efficient DMUs</th>
<th>$d_q$</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.08925</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-0.70655</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>0.07515</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>0.00731</td>
<td>3</td>
</tr>
<tr>
<td>21</td>
<td>0.00282</td>
<td>4</td>
</tr>
<tr>
<td>23</td>
<td>-0.00143</td>
<td>8</td>
</tr>
<tr>
<td>24</td>
<td>0.00269</td>
<td>5</td>
</tr>
<tr>
<td>25</td>
<td>-0.00058</td>
<td>7</td>
</tr>
<tr>
<td>26</td>
<td>-0.00039</td>
<td>6</td>
</tr>
<tr>
<td>27</td>
<td>-0.00668</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 4 shows the results of ranking with the proposed methods which are compared with other methods.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>1</th>
<th>2</th>
<th>6</th>
<th>8</th>
<th>21</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>AP (input oriented)</td>
<td>infeasible</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>MAJ (input oriented)</td>
<td>infeasible</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>8</td>
<td>6</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>AP (output oriented)</td>
<td>6</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>infeasible</td>
<td>infeasible</td>
<td>infeasible</td>
</tr>
<tr>
<td>MAJ (output oriented)</td>
<td>6</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>infeasible</td>
<td>infeasible</td>
<td>infeasible</td>
</tr>
<tr>
<td>Norm $L_1$</td>
<td>1</td>
<td>8</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>10</td>
<td>9</td>
<td>5</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Proposed method</td>
<td>1</td>
<td>10</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>8</td>
<td>5</td>
<td>7</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 4 shows that both AP (input oriented) and MAJ (input oriented) models in unit 1 are infeasible, AP method (output oriented) is infeasible in units 25, 26, 27 and MAJ method (output oriented) is infeasible in units 25, 26, 27, but the proposed method is always feasible. Zhu’s method is applied to units 2, 6, 8, 21, 23, 24, 25, 26, 27 (Table 5).
Larger $d_j$, shows better performance of $DMU_j$ in terms of the $r^{th}$ output and $t^{th}$ input compared with other $DMUs$. The unit 23 in the fifth ratio and the unit 24 in the seventh ratio are greater than unit 2, in norm $L_1$ method of ranking; it ranks unit 23 as 10, unit 24 as 9 and unit 2 as 8, but the proposed method ranks unit 23 as 8, unit 24 as 5 and unit 2 as 10. The unit 21 in the fifth and seventh ratios are greater than units 25, 26, 27 respectively, in norm $L_1$ ranking method, it ranks unit 21 as 7, unit 25 as 5, unit 26 as 6 and unit 27 as 4, but the proposed method ranks unit 21 as 4, unit 25 as 7, unit 26 as 6 and unit 27 as 9. The unit 24 in the seventh, fifth and eighth ratios is greater than units 25, 26 and 27 respectively, in norm $L_1$ ranking method, it ranks unit 24 as 9, unit 25 as 5, unit 26 as 6 and unit 27 as 4, but the proposed method ranks unit 24 as 5, unit 25 as 7, unit 26 as 6 and unit 27 as 9. The unit 26 in the fifth and eighth ratios is greater than unit 25 and unit 27 respectively, in norm $L_1$ ranking method, it ranks unit 26 as 6, unit 25 as 5 and unit 27 as 4, but the proposed method ranks unit 26 as 6, unit 25 as 7 and unit 27 as 9. The unit 25 in the ninth ratio is greater than unit 27, in norm $L_1$ ranking method, it ranks unit 25 as 5, unit 27 as 4, but the proposed method ranks unit 25 as 7, and unit 27 as 9. The unit 6 in the fifth ratio is greater than unit 8, in norm $L_1$ ranking method, it ranks unit 6 as 3, unit 8 as 2, but the proposed method ranks unit 6 as 2, unit 8 as 3. Thus, it can be said that the proposed method, in comparison with norm $L_1$ method, is the more accurate.

5. Conclusion

We introduced a new method for ranking efficient $DMUs$ based on TOPSIS and virtual $DMUs$. The comparison of the proposed method with other methods shows the proposed method is better than AP (input oriented), MAJ (input oriented), AP (output oriented), MAJ (output oriented) and norm $L_1$ methods. The proposed method is always feasible and simpler.
than other compared methods. As a forthcoming research, this study can be extended to the interval data.

References