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Toward Characterizing Solutions to Complex Programming Problems Involving Fuzzy Parameters in Constraints

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Abstract

The current study investigates to characterize the Complex Programming Problem (CPP) solution in a fuzzy environment. The paper is divided into two parts: 1) the first presents a Fuzzy Complex Programming Problem (F-CPP) with fuzzy complex constraints, and 2) the second presents the optimality criteria using the fuzzy complex cone. The CPP is suggested by involving fuzzy numbers in the constraints in parts. Using the $\alpha-$cut set concepts, the problem is converted into the $\alpha-$complex programming. A number of basic theorems with proofs are established concerning the basic results for the fuzzy complex set of solutions for the F-CPP, and the optimality criteria of the saddle point for F-CPP with fuzzy cones is derived.

Keywords: Complex programming, Fuzzy numbers, α-cut set, Kuhn-Tucker's optimality conditions, Saddle point, Cone, Duality.

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1 | Introduction

Complex Programming Problem (CPP) applications are found in [1]–[3]. In earlier works in the field of CPP, the majority of the authors considered only the real part of the objective function of the problem as the objective function of the problem, neglecting the imaginary part, and the corresponding constraints have been assumed as a cone in the complex space $\mathbb{C}^n$. However, in several applications of the real-world problem, the imaginary part involved in objective function especially plays a crucial role.

Mathematical programming in complex space has been originated by [1], where Farkas's theorem has generalized to the complex space. Hanson and Mond [4] have generalized Wolfe's duality optimization from real lines to complex numbers. Duca [5] formulated the vectorial optimization model involving complex numbers and derived the necessary and sufficient conditions at the point to be the efficient solution to the problem. Ferrero [6] considered the finite dimensional spaces using the separation arguments. In addition, the optimality conditions have been established in the real and imaginary parts of the objective function. Abrams [7] established sufficient conditions for optimal points of the objective function's real part, neglecting the imaginary part. Ben-Israel [8] introduced two theorems with proofs for equalities. Smart and Mond [9] have shown that the necessary conditions for optimality in polyhedral-cone-constrained nonlinear programming problems are sufficient with the special type of invexity hypothesis. In addition, they extended the duality results for a Wolfe-type dual. Youness and Elborolosy [10] formulated the optimization problem involving complex numbers. Malakooti [11] developed a complex method with interior search directions to solve linear and nonlinear programming problems.

One of the difficulties that emerged with the application of MP is that the parameters are not constants but uncertain. In fuzzy sets, as Zadeh [12] proposed, fuzzy numerical data is used as the fuzzy subsets of real lines, referred to as fuzzy numbers. Later, Dubois and Prade [13] studied the applications of mathematical operations on real lines to fuzzy numbers with the help of the fuzzification principle. The fuzzy nature of a goal programming problem was discussed by [14] and later followed by [15], [16], and many other authors working in that field. Tanaka et al. [17] introduced the concepts of fuzzy mathematical programming problems, following Bellman and Zadeh [18] and then by Tanaka and Asai [19].

Buckley [20] proposed the definition of fuzzy complex numbers following the concept of $\alpha-\text{cut}$ and investigated two special types depending on the forms $z = x + iy$ and $z = re^{i\gamma}$. Zhang and Xia [21] proposed two algorithms for solving complex quadratic mathematical programming problems with linear equality constraints and both an l_1 -norm and linear equality constraints. Zhang and Xia [3] proposed two complex-valued optimization solutions to constrained nonlinear programming problems of real functions in complex variables. Khalifa et al. [22] characterized the solution of complex nonlinear programming with interval-valued neutrosophic trapezoidal fuzzy parameters. Many researchers have developed the optimization model in complex spaces (for instance, Mond [23], Huang [24], and Elbrolosy [25]). Usha and Kumar [26] studied the queuing model using the fuzzy ranking method in a fuzzy environment. Khalifa et al. [22] characterized the solution of complex nonlinear programming with interval-valued neutrosophic trapezoidal fuzzy parameters. Qiu et al. [27] developed a fuzzy relation bi-level optimization model. They presented an application in the wireless communication station system. Khalifa and Kumar [28] recently applied fuzzy goal programming for nonlinear complex programming in a neutrosophic environment.

This paper is divided into two parts: The first part considers an Fuzzy Complex Programming Problem (F-CPP) with fuzzy complex constraints, and the second part presents the optimality criteria using the fuzzy complex cone. The CPP is considered in all the two parts by involving fuzzy numbers in the constraints. Using the $\alpha-$cut set concepts, the problem is converted into the corresponding $\alpha-$complex programming. Some basic theorems with proofs are established concerning the basic results for the fuzzy complex set of solutions for the F-CPPs, and the optimality criteria of the saddle point for F-CPP with fuzzy cones is derived.

The remaining structure of this study is outlined below:

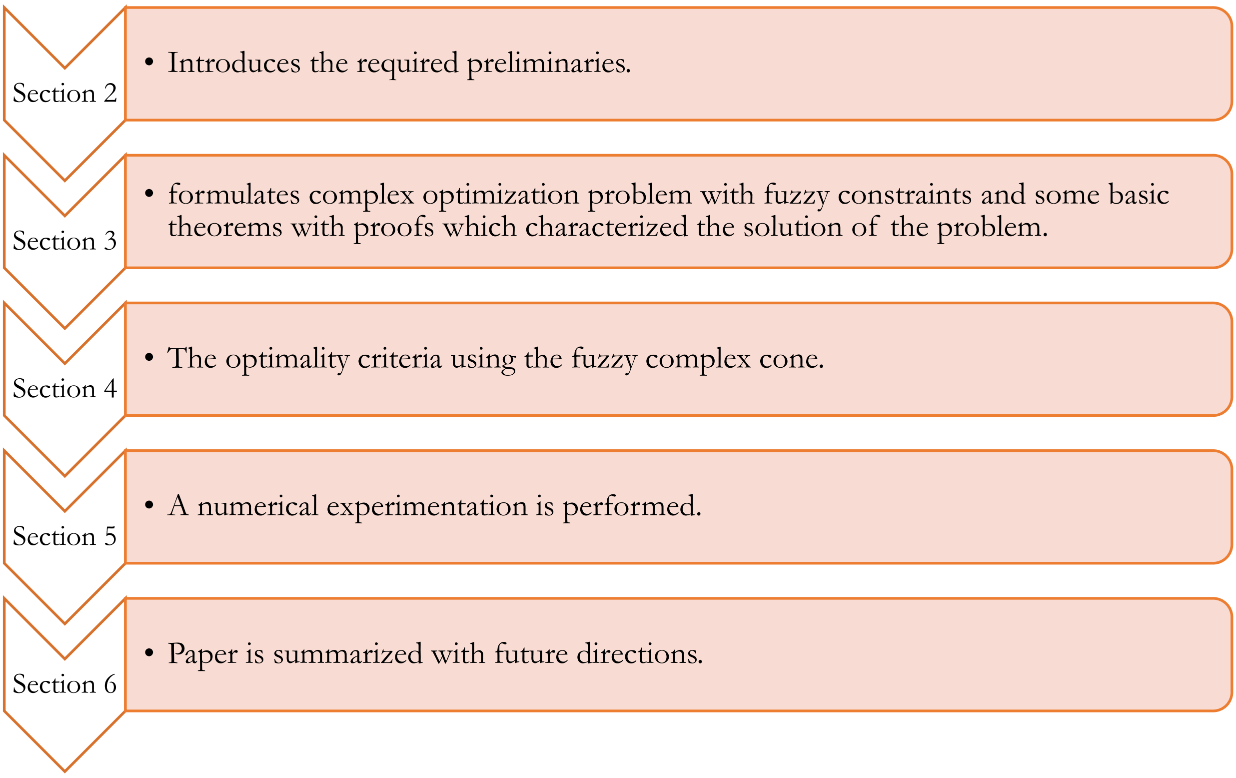

Fig. 1. Research method.

2 | Preliminaries

Represent $\Re = (-\infty, \infty)$ as real line, and $z = \{x + iy : x, y \in \Re,\ i = \sqrt{-1}\}$ as a complex number, $\mathbb{C}$ = field of complex numbers.

Definition 1 ([20]). A map $\tilde{Z}\colon\ \mathbb{C} \to [0,1]$ is referred to as a fuzzy complex set, $\mu_{\tilde{Z}}(z)$ is said to be a membership function of $\tilde{Z}$ for z, $F(\mathbb{C}) = \{\tilde{Z}: \tilde{Z}\colon\ \mathbb{C} \to [0,1]\}$ represents all fuzzy complex sets on $\mathbb{C}$.

Definition 2. The α −cut set of a complex set $\tilde{Z}$, designated by $\tilde{Z}_\alpha$, can be written as follows:

$\tilde{Z}_\alpha = \{z = x + i\,y \in \mathbb{C}: \mu_{\tilde{Z}}(z) \geq \alpha\}$.

Definition 3 ([20]). Supp $\tilde{Z} = \{z = x + i\,y \in \mathbb{C}: \mu_{\tilde{Z}}(z) \geq 0\}$ is referred to as the support of $\tilde{Z}$.

Definition 4 ([20]).

I. $\tilde{Z} \in F(\mathbb{C})$ is convex fuzzy complex set on $\mathbb{C}$, iff for all $\alpha \in [0,1]$, $\tilde{Z}_\alpha$ is a convex complex set.

II. $\tilde{Z} \in F(\mathbb{C})$ is a closed fuzzy complex set on $\mathbb{C}$, iff for $\alpha \in [0,1]$, $\tilde{Z}_\alpha$ is a closed complex set.

III. If $\tilde{Z} \in F(\mathbb{C})$, Supp $\tilde{Z}$ is bounded set on $\mathbb{C}$, iff for $\alpha \in [0,1]$, $\tilde{Z}_\alpha$ is bounded on $\mathbb{C}$.

IV. $\tilde{Z} \in F(\mathbb{C})$ is normal fuzzy complex set on $\mathbb{C}$, iff for $\alpha \in [0,1]$, $\{z \in \mathbb{C}: \mu_{\tilde{Z}}(z) = 1\} \neq \emptyset$.

Definition 5. A normal convex fuzzy complex set on $\mathbb{C}$ is referred to as a fuzzy complex number.

Definition 6 ([5], [29]). Let us consider that $\phi \neq H \subset \mathbb{C}^n$.

I. H is a cone, provided that for each $z \in H$ and each $\alpha \in\ ]0, \infty[$, we have $\alpha\, z \in H$.

II. H is a convex cone, when it is both – convex as well as a cone.

III. Intersection of all convex cones in $\mathbb{C}^n$ containing the set H is a convex cone spanned by H, and it is designated by con (H).

Definition 7 ([5]). A polyhedral cone S in $\mathbb{C}^n$ is a convex cone generated by finitely many vectors, that is, a set of the form $S\,\Re_+^k = \{Ax: x \in \Re_+^k\}$, for some positive integer k and $A \in \mathbb{C}^{n\times k}$.

3 | Problem Definition (Part I)

Let us recall a problem in complex space as follows:

$$\begin{aligned}&\min \operatorname{Re} f(z),\\ &\text{s.t.}\\ &z \in \widetilde{M},\end{aligned} \tag{1}$$

where $\widetilde{M}$ referrers to a fuzzy complex non-empty subset of $\mathbb{C}^n$ (i.e., $\emptyset \neq \widetilde{M} \subset \mathbb{C}^n$), and $f: \widetilde{M} \to \mathbb{C}$ is a function of complex variable z.

Problem (1) is transformed to its crisp counterpart as below:

$$\begin{aligned}&\min \operatorname{Re} f(z),\\ &\text{s.t.}\\ &z \in \mu_{\widetilde{M}}(z),\end{aligned} \tag{2}$$

where $\mu_{\widetilde{M}}(z) = \mu_{\widetilde{M}}(x,y) = \min\left(\mu_{\widetilde{M}}(x), \mu_{\widetilde{M}}(y)\right), \mu_{\widetilde{M}}(x): \Re^n \to [0,1], x \in \Re^n;\ \mu_{\widetilde{M}}(y): \Re^n \to [0,1], y \in \Re^n$.

Definition 8. The fuzzy feasible point $z^\circ \in \widetilde{M}$ with the membership function $\mu_{\widetilde{M}}(z^\circ) = \mu_{\widetilde{M}}(x^\circ, y^\circ) = \min\left(\mu_{\widetilde{M}}(x^\circ), \mu_{\widetilde{M}}(y^\circ)\right), z^\circ = x^\circ + i\,y^\circ$ is a fuzzy optimal solution to the *Problem (2)* when $\operatorname{Re} f(z^\circ) = \min\left(\operatorname{Re} f(z): z \in \widetilde{M} \text{ with membership } \mu_{\widetilde{M}}(z)\right)$.

Theorem 1. Let $\widetilde{M}$ be a non-empty fuzzy complex subset of $\mathbb{C}^n$ and $f: \widetilde{M} \to \mathbb{C}$ be a function of quasi-convex real part on $\widetilde{M}$ relative to $\Re_+ = [0, \infty[$. Then, $\widetilde{M}$ is convex.

Proof: Let z_1 and z_2 be two solutions of *Problem (2)*, then $z_1, z_2 \in \widetilde{M}$ with $\mu_{\widetilde{M}}(z_1), \mu_{\widetilde{M}}(z_2)$, and $\operatorname{Re} f(z_1) = \operatorname{Re} f(z_2) = \min\left(\operatorname{Re} f(z): z \in \widetilde{M}\right)$ and therefore

$$0 = \operatorname{Re} f(z_1) - \operatorname{Re} f(z_2) \in \Re_+. \tag{3}$$

Since $\widetilde{M}$ is a fuzzy complex set, then for $0 \leq \zeta \leq 1 \Rightarrow z_\zeta = \zeta z_1 + (1-\zeta) z_2 \in \widetilde{M}$ with $\mu_{\widetilde{M}}(\zeta z_1 + (1-\zeta) z_2) \geq \min(\mu_{\widetilde{M}}(z_1), \mu_{\widetilde{M}}(z_2))$. From the assumption of the *Theorem 1* and from *Eq. (3)*, we have

$$\operatorname{Re} f(z_1) - \operatorname{Re} f(z_2) \in \Re_+. \tag{4}$$

Or equivalently, $\operatorname{Re} f(z_1) \leq \operatorname{Re} f(z_2)$. Thus, z_ζ is a convex set.

Theorem 2. Let us consider that $\phi \neq \widetilde{M} \subset \mathbb{C}^n$, and $f: \widetilde{M} \to \mathbb{C}$ be a function and z_1 be a solution to *Problem (1)*. If f has a strictly convex real part of z_1 following $\Re_+$, then z_1 is the unique solution to the *Problem (1)*.

Proof: Let z_2 be another solution to the *Problem (1)*, $z_1 \neq z_2$, then $z_1, z_2 \in \widetilde{M}$ with membership function $\mu_{\widetilde{M}}(z_1) = \mu_{\widetilde{M}}(x_1, y_1) = \min(\mu_{\widetilde{M}}(x_1), \mu_{\widetilde{M}}(y_1))$, $\mu_{\widetilde{M}}(z_2) = \mu_{\widetilde{M}}(x_2, y_2) = \min(\mu_{\widetilde{M}}(x_2), \mu_{\widetilde{M}}(y_2))$, with $\mu_{\widetilde{M}}(x_1): \Re^n \to [0,1]$. Since $z_1 = x_1 + i\,y_1, z_1 \in \widetilde{M} \subset \mathbb{C}^n, \operatorname{Re}(z_1) = \operatorname{Re}(z_2) \min \operatorname{Re}\left\{f(z): z \in \widetilde{M}\right\}$, this leads to

$$\operatorname{Re} f(z_1) - \operatorname{Re}(z_2) = 0. \tag{5}$$

Let $0 \leq \zeta \leq 1$. Since, $\widetilde{M}$ is convex in $\mathbb{C}^n$, then $z_\zeta = (1-\zeta) z_1 + \zeta z_2 \in \widetilde{M}$ with membership $\mu_{\widetilde{M}}\left((1-\zeta) z_1 + \zeta z_2\right) \geq \min(\mu_{\widetilde{M}}(z_1), \mu_{\widetilde{M}}(z_1))$. Moreover, since f has a strictly convex real part at z_1 with respect to $\Re_+$, it follows that

$$\operatorname{Re}\left((1-\zeta) f(z_1) + \zeta f(z_2) - f(z_\zeta)\right) > 0, \tag{6}$$

Therefore,

$$\operatorname{Re} f(z_\zeta) < \operatorname{Re}\left((1-\zeta)f(z_1) + \zeta f(z_2)\right) < \operatorname{Re} f(z_1) - \zeta \operatorname{Re} f(z_1) + \zeta \operatorname{Re} f(z_2) = \operatorname{Re} f(z_1). \quad \mathbf{(7)}$$

This contradicts that z_1 is a solution to *Problem (1)*.

Definition 9. The point $z^{\circ} \in \widetilde{M}$ with membership $\mu_{\widetilde{M}}(z^{\circ}) = \mu_{\widetilde{M}}(x^{\circ}, y^{\circ}) = \min(\mu_{\widetilde{M}}(x^{\circ}), \mu_{\widetilde{M}}(y^{\circ})$ is a local solution to *Problem (1)* if there is a neighbourhood $\tilde{u}$ of the point z° so that $\operatorname{Re} f(z^{\circ}) = \min\{\operatorname{Re} f(z) : z \in \widetilde{M} \cap \tilde{u}\}$, with membership function $\mu_{\widetilde{M}\cap\tilde{u}} = \min(\mu_{\widetilde{M}}(z)), \mu_{\tilde{u}}(z))$.

Theorem 3. Let us consider that $\phi \neq \widetilde{M} \subset \mathbb{C}^n$, and $f : \widetilde{M} \to \mathbb{C}$ be a function with concave real part on $\widetilde{M}$ relative to $\mathfrak{R}_+$, under the assumption that its real part is not a constant, if z° is a solution to *Problem (1)*, then z° is a member of the set representing the boundary of $\widetilde{M}$.

Proof: If $\operatorname{int}(\widetilde{M}) = \emptyset$, then z° belongs to the boundary of $\widetilde{M}$. Assume the case when $\operatorname{int}(\widetilde{M}) \neq \emptyset$. Since $\operatorname{Re} f(z)$ is not constant, it follows that there exists $z_1 \in \widetilde{M}$ with the membership function $\mu_{\widetilde{M}}(z_1)$ such that

$$\operatorname{Re} f(z^{\circ}) < \operatorname{Re} f(z_1). \quad \mathbf{(8)}$$

Now, let $z \in \operatorname{int}(\widetilde{M})$, then there exists a real number $\varepsilon > 0$ so that $\widetilde{B}(z,\varepsilon) \subseteq \widetilde{M}$ with $\mu_{\widetilde{B}(z,\varepsilon)} < \mu_{\widetilde{M}}(z)$. Let us denote

$$\zeta = \frac{\|z - z_1\| + 1}{\|z - z_1\| + 1 + \varepsilon}, 0 < \zeta < 1.$$

Let $w = \frac{1}{\zeta} z + \left(1 - \frac{1}{\zeta}\right) z_1$. Since

$$\|z - w\| = \left\| z - \frac{1}{\zeta} z - z_1 + \frac{1}{\zeta} z_1 \right\|$$

$$= \left\| z\left(1 - \frac{1}{\zeta}\right) - \left(1 - \frac{1}{\zeta}\right) z_1 \right\|$$

$$= \left\| \left(1 - \frac{1}{\zeta}\right)(z - z_1) \right\|$$

$$= \left|1 - \frac{1}{\zeta}\right| \|z - z_1\|$$

$$= \left|1 - \frac{\|z - z_1\| + 1 + \varepsilon}{\|z - z_1\| + 1}\right| . \|z - z_1\|$$

$$= \left|1 - \frac{\|z - z_1\| + 1}{\|z - z_1\| + 1} - \frac{\varepsilon}{\|z - z_1\| + 1}\right| . \|z - z_1\|$$

$$= \left|\frac{-\varepsilon}{\|z - z_1\| + 1}\right| . \|z - z_1\| < \varepsilon$$

$$= \frac{\varepsilon}{\|z - z_1\| + 1} . \|z - z_1\| < \varepsilon.$$

It follows that, $w \in \widetilde{B}(z,\varepsilon)$, and $w \in \widetilde{M}$ wit memberships $\mu_{\widetilde{B}(z,\varepsilon)}(w) \leq \mu_{\widetilde{M}}(w)$. Then, we deduce that $\operatorname{Re} f(z^{\circ}) \leq \operatorname{Re}(w)$, referring to the equation

$$w = \frac{1}{\zeta} z + \left(1 - \frac{1}{\zeta}\right) z_1 \Rightarrow \frac{1}{\zeta} z = w - z_1 + \frac{1}{\zeta} z_1 \Rightarrow \frac{1}{\zeta} z = w + \left(\frac{1}{\zeta} - 1\right) z_1.$$

Hence, we obtain $z = \zeta w + z_1 - \zeta z_1$, i.e., $z = \zeta w + (1 - \zeta) z_1$, since $z_1 \in \widetilde{M}, w \in \widetilde{B}(z,\varepsilon) \subseteq \widetilde{M}$, $0 < \zeta < 1$, then $z \in \widetilde{M}$ with the following membership function

$$\mu_{\widetilde{M}} \geq \min(\mu_{\widetilde{M}}(w), \mu_{\widetilde{M}}(z_1)).$$

Since $f(z)$ has a concave real part on $\widetilde{M}$ relative to $\mathfrak{R}_+$, we have

$$\begin{aligned}&\operatorname{Re} f(z)=\operatorname{Re} f\big((1-\zeta)z_1+\zeta w\big)\\&\geq(1-\zeta)\operatorname{Re} f(z_1)+\zeta \operatorname{Re} f(w)\\&>(1-\zeta)\operatorname{Re} f\big(z^{\circ}\big)+\zeta \operatorname{Re} f\big(z^{\circ}\big)=\operatorname{Re} f\big(z^{\circ}\big).\end{aligned}$$

If $z \in \operatorname{int}(\widetilde{M})$. Then

$$\operatorname{Re} f\big(z^{\circ}\big)<\operatorname{Re} f(z). \tag{9}$$

Thus, $\operatorname{Re} f$ cannot attain its minimum at an interior point of $\widetilde{M}$.

Remark 1: Let z° be a solution of *Problem (1)*. Then, z° is a local solution to *Problem (1)*. But the converse does not generally hold.

Theorem 4. Let z° be a non-empty fuzzy subset of $\mathbb{C}^n$ with membership function $\mu_{\widetilde{M}}(z)=\mu_{\widetilde{M}}(x,y)=\min(\mu_{\widetilde{M}}(x),\mu_{\widetilde{M}}(y))$, where $\mu_{\widetilde{M}}(x):\mathfrak{R}^n\to[0,1]$, $\mu_{\widetilde{M}}(y):\mathfrak{R}^n\to[0,1]$, $x,y\in\mathfrak{R}^n$, and $f:\widetilde{M}\to\mathbb{C}$ be a function with a convex real part on $\widetilde{M}$ relative to $\mathbb{R}_+$. If z° is a local solution of *Problem (1)*, then z° is a solution of *Problem (1)* with membership $\mu_{\widetilde{M}}(z^{\circ})=\mu_{\widetilde{M}}(x^{\circ},y^{\circ})=\min(\mu_{\widetilde{M}}(x^{\circ}),\mu_{\widetilde{M}}(y^{\circ}))$, $\mu_{\widetilde{M}}(x^{\circ}):\mathfrak{R}^n\to[0,1]$, $\mu_{\widetilde{M}}(y^{\circ}):\mathfrak{R}^n\to[0,1]$, $x^{\circ},y^{\circ}\in\mathfrak{R}^n$.

Proof: Let z° be a local solution to *Problem (1)*. Then there is a $\varepsilon\in\mathfrak{R}$, $\varepsilon>0$, so that

$$\operatorname{Re} f\big(z^{\circ}\big)<\operatorname{Re} f(z),\ \text{ for all }\ z\in\widetilde{M}\cap\mu_{\widetilde{B}(z^{\circ},\varepsilon)}, \tag{10}$$

with membership function

$$\mu_{\widetilde{M}}(z)=\min(\mu_{\widetilde{M}}(x^{\circ}),\mu_{\widetilde{M}}(y^{\circ})),\ \mu_{\widetilde{B}(z^{\circ},\varepsilon)}(z)=\min\big(\mu_{\widetilde{M}}(z),\mu_{\widetilde{B}(z^{\circ},\varepsilon)}\big).$$

Let us assume that there exists a point z_1 with the property that

$$\operatorname{Re} f(z_1)<\operatorname{Re} f(z^{\circ}). \tag{11}$$

It can be observed that $z_1=z^{\circ}$. Since, $\widetilde{M}$ is a fuzzy convex set, it follows that $z_\zeta=(1-\zeta)z^{\circ}+\zeta z^{\circ}\in\widetilde{M}$; for all $0\leq\zeta\leq1$, and membership $\mu_{\widetilde{M}}\big((1-\zeta)z^{\circ}+\zeta z^{\circ}\big)\geq\min(\mu_{\widetilde{M}}(z^{\circ}),\mu_{\widetilde{M}}(z^1))$, let us choose $0\leq\zeta\leq1$ such that $0<\zeta<\frac{\varepsilon}{\|z^{\circ}-z_1\|}$.

Then, $\|z_\zeta-z^{\circ}\|=\|(1-\zeta)z^{\circ}+\zeta z_1-z^{\circ}\|=\zeta\,\|z^{\circ}-z_1\|\leq\varepsilon$, and thus $z_\zeta\in\widetilde{M}\cap\widetilde{B}(z^{\circ},\varepsilon)$, with membership $\mu_{\widetilde{M}\cap\widetilde{B}(z^{\circ},\varepsilon)}=\min\big(\mu_{\widetilde{M}(z_\zeta)},\mu_{\widetilde{B}(z^{\circ},\varepsilon)}(z_\zeta)\big)$. Hence, we obtain

$$\operatorname{Re} f\big(z^{\circ}\big)<\operatorname{Re} f\big(z_\zeta\big).$$

Moreover, since the function f has a convex real part on $\widetilde{M}$ relative to $\mathbb{R}_+$, it follows that

$$\begin{aligned}&\operatorname{Re} f(z_1)<\operatorname{Re} f\big(z^{\circ}\big),\\&\operatorname{Re} f\big(z_\zeta\big)<(1-\zeta)\operatorname{Re} f\big(z^{\circ}\big)+\zeta\operatorname{Re} f(z_1)<(1-\zeta)\operatorname{Re} f\big(z^{\circ}\big)+\zeta\operatorname{Re} f\big(z^{\circ}\big)\Rightarrow\operatorname{Re} f(z_1)<\\&\operatorname{Re} f\big(z^{\circ}\big).\end{aligned} \tag{12}$$

Which contradicts *Eq. (11)* and so there is no other solution for the *Problem (1)*, and so the local solution z° is a global solution to *Problem (1)*.

4 | Problem Statement (Part II)

Let us recall an optimization problem as below:

$$\begin{aligned}&\min \operatorname{Re} f(z),\\&\text{s.t.}\\&z\in\widetilde{X},\\&g(z)\in\widetilde{U},\end{aligned} \tag{13}$$

where $\emptyset \neq \tilde{X} \subset \mathbb{C}^n, \emptyset \neq \tilde{U}$, with memberships $\mu_{\tilde{X}}(z) = \mu_{\tilde{X}}(x,y) = \min(\mu_{\tilde{X}}(x), \mu_{\tilde{X}}(y))$, $\mu_{\tilde{X}}(x): \Re^n \to [0,1], x \in \Re^n, \mu_{\tilde{X}}(y): \Re^n \to [0,1], y \in \Re^n, \mu_{\tilde{U}}(g(z)) = \mu_{\tilde{U}}(u(x,y), v(x,y)) = \min(\mu_{\tilde{U}}(u), \mu_{\tilde{U}}(v)), z = x + i\,y, g(x,y) = u(x,y) + i\,v(x,y)$, $\mu_{\tilde{U}}(u): \Re^m \to [0,1], u \in \Re^m, \mu_{\tilde{U}}(v): \Re^m \to [0,1], v \in \mathbb{R}^m$, and $f: \tilde{X} \to \mathbb{C}, g: \tilde{X} \to \mathbb{C}^m$ are two functions. For $r \in \mathbb{R}$, and *Problem (13)*, we define the function as follows:

$$\varphi_r(z,v) = r\,f(z) - \langle g(z), v\rangle), \text{for all } (z,v) \in \tilde{X} \times \tilde{U}_*, \mu_{\tilde{X}\times\tilde{U}_*}(z,v) = \mu_{\tilde{X}}(z) \times \mu_{\tilde{U}_*}(v). \quad \textbf{(14)}$$

Definition 10. $Y \subset \mathbb{C}^n$ referrers to fuzzy cone with membership function $U: Y \to [0,1]$, if:

I. $U(0) = 1$.

II. $U(\zeta z) \leq U(z)$; for all $z \in Y, \zeta \geq 0, z = x + i\,y, U(z) = U(x,y) = \min(U(x), U(y)), U(x): \mathbb{R}^n \to [0,1], U(y): \mathbb{R}^n \to [0,1]$.

Remark 2: In the case of the complex cone U generalized by infinite vectors, a fuzzy complex cone is termed a fuzzy polyhedral cone.

Example 1. Let $Y \subset \mathbb{C}^n$ be the space of all complex numbers, the function U, defined by

$$U(z) = U(x,y) = \begin{cases} 0, & \text{if } x < 0 \lor y < 0, \\ \dfrac{y}{x}, & \text{if } x > 0, y > 0 \land y < x, \\ 1, & \text{if } x \geq 0, y \geq 0 \land y \geq x, \end{cases}$$

is an example of a fuzzy complex cone in $\mathbb{C}$.

Theorem 5. Suppose $\emptyset \neq \tilde{X} \subset \mathbb{C}^n$, $\tilde{U}$ is a fuzzy polyhedral cone in $\mathbb{C}^m$ with non-empty interior $f: \tilde{X} \to \mathbb{C}$ representing a function with a convex real part on $\tilde{X}$ relative to $\Re_+$, and $g: \tilde{X} \to \mathbb{C}^m$ be a concave function on $\tilde{X}$ with respect to $\tilde{U}$. If z° is a solution of *Problem (13)*, then there is a $r_{\circ} \in \Re, v_{\circ} \in \tilde{U}_*$ with membership

$$\mu_{\tilde{U}_*}(v_{\circ}) = \mu_{\tilde{U}_*}(x_{v_{\circ}}, y_{v_{\circ}}) = \min(\mu_{\tilde{U}_*}(x_{v_{\circ}}), \mu_{\tilde{U}_*}(y_{v_{\circ}})). \quad \textbf{(15)}$$

$$\operatorname{Re}\langle g(z^{\circ}), v_{\circ}\rangle = 0. \quad \textbf{(16)}$$

$$\operatorname{Re}\varphi_{r_{\circ}}(z^{\circ}, v) < \operatorname{Re}\varphi_{r_{\circ}}(z^{\circ}, v_{\circ}) \leq \operatorname{Re}\varphi_{r_{\circ}}(z, v_{\circ}), \quad \text{for all } (z^{\circ}, v_{\circ}) \in \tilde{X} \times \tilde{U}, \quad \textbf{(17)}$$

where $\varphi_{r_{\circ}}(z,v) = r\,f(z) - \langle g(z), v\rangle)$; for all$(z,v) \in \tilde{X} \times \tilde{U}_*$.

Proof: Since z° is a solution to *Problem (13)*, therefore the following system:

$$\begin{aligned} &\operatorname{Re}\left(f(z) - f(z^{\circ})\right) < 0, \\ &z \in \tilde{X}, \\ &g(z) \in \tilde{U}, \end{aligned} \quad \textbf{(18)}$$

is inconsistent.

Denote $\widetilde{\mathbb{C}\Re_+}$ be the closed fuzzy convex cone, which is defined by $\mu_{\widetilde{\mathbb{C}\Re_+}}(x,y) = \min\left(\mu_{\widetilde{\mathbb{C}\Re_+}}(x), \mu_{\widetilde{\mathbb{C}\Re_+}}(y)\right)$, $\mu_{\widetilde{\mathbb{C}\Re_+}}(x): \Re \to [0,1], \mu_{\widetilde{\mathbb{C}\Re_+}}(y): \Re \to [0,1]$, $v = x + i\,y$, for all $x, y \in \Re$.

Let us rewrite the *System (18)* as expressed in the following form:

$$\begin{aligned} &f(z^{\circ}) - f(z) \in \operatorname{int}\left(\widetilde{\mathbb{C}\Re_+}\right), \\ &z \in \tilde{X}, \\ &g(z) \in \tilde{U}, \end{aligned} \quad \textbf{(19)}$$

Since the *System (19)* is consistent, it follows that there exists $r_{\circ} \in \Re, v_{\circ} \in \mathbb{C}^m$ such that

$$r_{\circ} \in (\widetilde{\mathbb{C}\Re_+})_* = \Re_+, v_{\circ} \in \mathbb{C}^m, (r_{\circ}, v_{\circ}) \neq (0,0). \quad \textbf{(20)}$$

$$\operatorname{Re}\left(\langle f(z^{\circ}) - f(z), r_{\circ}\rangle + \langle g(z), v_{\circ}\rangle\right) \leq 0, \quad \text{for all } z \in \tilde{X}. \quad \textbf{(21)}$$

Let us choose $z = z^{\circ} \in X$, we have

$$\mathrm{Re}\,\langle g(z^{\circ}), v_{\circ}\rangle \leq 0. \tag{22}$$

Because $v_{\circ} \in U_{*}$ and $g(z^{\circ}) \in \widetilde{U}$, we get $\mathrm{Re}\,\langle g(z^{\circ}), v_{\circ}\rangle \geq 0$, and therefore, $\mathrm{Re}\,\langle g(z^{\circ}), v_{\circ}\rangle = 0$.

From *Eq. (16)* and *Inequality (21)* and the fact that z° is a solution of *Problem (13)*, it follows that

$$\mathrm{Re}\big(r_{\circ}f(z^{\circ}) - \langle g(z^{\circ}), v_{\circ}\rangle\big) \leq \mathrm{Re}\,(r_{\circ}f(z) - \langle g(z), v_{\circ}\rangle),\ \text{ for all } z \in \widetilde{X}, \tag{23}$$

which is the same as in *Inequality (18)*.

In order to choose that the *Inequality (17)* is satisfied, from $g(z^{\circ}) \in \widetilde{U}$ with membership

$$\mu_{\widetilde{X}}\left(g(z^{\circ})\right) = \min\left(\mu_{\widetilde{X})}u(x^{\circ}, y^{\circ}), \mu_{\widetilde{X}}v(x^{\circ}, y^{\circ})\right).$$

We get $\mathrm{Re}\,\langle g(z^{\circ}), v\rangle \geq 0;$ for all $v \in \widetilde{U}_{*}$, and therefore $\mathrm{Re}\,(r_{\circ}f(z^{\circ}) - \langle g(z^{\circ}), v_{\circ}\rangle) \leq \mathrm{Re}\,(r_{\circ}f(z^{\circ}) - \langle g(z^{\circ}), v_{\circ}\rangle);$ for all $v \in \widetilde{U}_{*}$, because of $\mathrm{Re}\langle g(z^{\circ}), v_{\circ}\rangle = 0$.

5 | Concluding Remarks

In the current study, we introduced some results for optimization in complex space with fuzzy complex set in the constraints, and also, the F-CPP with fuzzy complex cone in the constraint has been introduced. Some basic theorems that characterized the problem's solution have been stated with proof. There are many problems and research points to be investigated in the field of fuzzy complex MP problems; some of these points are as follows:

I. Study of fuzzy complex linear programming problem considering the real and imaginary parts.

II. Study of fuzzy complex fractional programming problems in both single-objective and multi-objective functions.

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