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Technical Analysis of Petrochemical Industries of Iran Using a Network Data Envelopment Analysis Model

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Abstract

The data envelopment analysis method is commonly used to measure efficiency. An estimate of the relative efficiency of this model is derived by calculating the ratio between inputs and outputs. Data envelopment analysis models can also be applied to network structures due to the extension of these models. Supply Chain Management (SCM) is a novel approach that governed production management in recent years. In complex and dynamic environments, the petrochemical industry requires an investigation system similar to those used by other organizations to inform about its activity's desirability, especially in complex and dynamic environments. This research focused on the petrochemical company supply chain. Laboratory studies, experts, and visits to petrochemical sites were used to identify production processes and determine indicators. After that, they were evaluated with an envelopment model and a coefficient corresponding to the identified petrochemical supply chain structure. The aggregate and componentwise efficiency of the studied units in petrochemical were also examined from 2016 to 2019.

Keywords: Performance evaluation, Network data envelopment analysis, Aggregate efficiency, Componentwise efficiency.

1 | Introduction

The data envelopment analysis method is commonly used to measure efficiency. An estimate of the relative efficiency of this model is derived by calculating the ratio between inputs and outputs [1]. After pioneering work of Charnes and Cooper [2] many scholars and researchers entered fuzzy set theory in DEA [2]-[4]. For instance, Bagherzadeh et al. [6] proposed a novel ranking method for DMUs based on fuzzy DEA. Nojehdehi et al. [7] proposed an approach to measure the production possibility based on fuzzy efficient frontier in DEA. In real world problem production systems have a network structure and the output of each stage is used as an input for the next stage, the data envelopment analysis network method is used to measure the efficiency of all model's components [8]. Therefore, unlike classic data envelopment analysis models, it helps to model organization and measure the efficiency of model components [9]. A significant challenge in the development of performance evaluation based on data envelopment analysis is distinguishing the model validation from a wide range of input and output indices [10].

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Based on Yang [11], whereas early DEA researches concentrated mostly on theoretical and methodological progrecess, the number of DEA research incorporating real-world applications has grown. Several network structures can be found in current systems, including series, parallel, and mixed structures [10]-[14]. The importance of each dimension of the network model should also be considered [17]. In this study, the main objective is to select a proper model by considering proper variables and defining the correct relationship between model dimensions, in addition to considering the right weight for each dimension of data envelopment analysis to assess the efficiency of different units in petrochemical companies and to analyze progress and regression in units. A crucial issue in evaluating the petrochemical industry's performance is the activity nature and the dividing their business. Data envelopment analysis is a comprehensive procedure that entered the petrochemical industry to study performance evaluation, and it still was not accepted by managers. The popularity of this model is due to the existence of multiple inputs and outputs in this model and its proportionality in the study of nonlinear equations in analyses. In the absence of consideration for sub-processes, a superficial assessment of performance was conducted. A few studies divided overall efficiency into partial efficiencies to analyze subsidiary processes and resources of inefficiency [18]. Therefore, the network data envelopment analysis method specifies the efficiency of the entire system and services provision process and calculates the efficiency of each part of the model. In the petrochemical industry, it allows managers to make strategic decisions to enhance each sub-process.

Many of the Decision-Making Units (DMUs) have more than one stage. By evaluating the performance of these units using data envelopment analysis, the entire DMU cannot be viewed as a black box. Rather, the internal equations should also be taken into consideration. Different methods were presented to study the efficiency of multistage units. In organizations, it is particularly challenging to calculate the efficiency of sub-sets that have a cause-and-effect relationship. Furthermore, the time factor affects their performance to a great extent. As a result, organizational analysis requires developing a plan based on dynamic models, considering the time factor. Supply chains are among units that have multiple stages, and reversible factors exist in some of them. Thus, providing models to evaluate the efficiency of multistage units in the presence of reversible factors is crucial. The main question of this research is how to create a mathematical model in a network to measure the performance of an organization so that the overall and component functions can be presented. This study evaluates some petrochemicals in the country based on their information modeled at three levels. A review of data envelopment analysis and supply chain introductions was the focus of the second section. A data envelopment analysis model is presented in Section 3 for evaluating the aggregate and componentwise efficiency of chains within the petrochemical industry. Section 4 presents a functional example and demonstrates how the models are implemented. Section 5 includes the conclusion and suggestions.

2 | Introductions of Data Envelopment Analysis and Supply Chain

2.1 | Concepts and Fundamentals of DEA

Assume a unit that consumes the X input and creates the Y output. The relative efficiency is defined as below:

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{Y}{X}. \tag{1}$$

This definition is practical when the DMU has one input and one output. It is assumed that the output is equal to y for specific DMUs of the global standard. If the DMU consumes one unit of input and produces y_o units of output, the absolute efficiency will be as below:

$$\frac{y_\circ}{y^*}. \tag{2}$$

The reasons for using relative efficiency in the performance evaluation of DMUs are:

First, in developing countries such as Iran, real unit performance often falls short of international standards, and no method can be presented to enable units to reach the standard level, or if presented, it would cause disappointment. Second, there are no standards for most organizations, and considering international standards is not reasonable for organizations.

Assume that the j[th] DMUs consume the xj input and create yj output. The relative efficiency for the p[th] unit, which is shown with REp, is defined below:

$$RE_p = \frac{\frac{y_p}{x_p}}{\operatorname{Max}\left\{\frac{y_j}{x_j} : j=1,\cdots,n\right\}}. \tag{3}$$

A DMU is a unit that receives the input vector, such as (x1,..., xm), to create the output vector, such as (y1,..., ys). A congruous DMU consists of units with similar performance and create similar outputs by receiving similar inputs. For example, branches of a bank are congruous units.

Consider a DMU that consumes an input vector of (x1, ..., xm) to create the output vector of (y1..., ys). The efficiency of such a unit is defined below:

$$\text{Efficiency} = \frac{u_1 y_1 + \cdots + u_s y_s}{v_1 x_1 + \cdots + v_m x_m}, \tag{4}$$

where ur is the price of the rth output, i.e. yr (r=1,..., s), and vi is the price of xi (i=1,...,m). This efficiency is known as economic efficiency. The x vector is dominant to the y vector if and only if $X \geq Y$ and $X \neq Y$, in which one can say that the Y vector has been conqured by the X vector.

Assume that we have n DMU, and each DMUj (j=1,..., n) uses m input of Xij (i=1,...,m) to create S output Yrj(r=1,..., s). DEA calculated the performance for DMUj as below:

$$h_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}, \tag{5}$$

where vi (i=1,…,m) and ur (r= 1,…,s) are the weight of the relative input and output of DMUj. Weights in *Eq. (5)* are determined by the below programming problem:

$$\begin{aligned} & h_0 = \max\ h_0, \\ & \text{s.t.} \\ & h_j \leq 1,\ j=1,2,\ldots,n, \\ & v_i, u_r \geq 0. \end{aligned} \tag{6}$$

The CCR in *Model (7)* is known to have an input orientaion in the envelope form.

$$\begin{aligned} & \operatorname{Min} \quad \theta \\ & \text{s.t.} \\ & \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \qquad i=1,2,\ldots,m, \\ & \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \qquad r=1,2,\ldots,s, \\ & \quad \lambda_j \geq 0, \qquad j=1,2,\ldots,n. \end{aligned} \tag{7}$$

Note that the model is always feasible and $0 < \theta^* \leq 1.$

If $\theta^* = 1$, MU is practical; otherwise, it is not. The dual envelopment form, known as the multiplication form, is as below:

$$\begin{aligned}
&\text{Max} \ \sum_{r=1}^{s} u_r y_{r\circ}, \\
&\text{s.t.} \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{s} v_i x_{ij} \leq \circ, \quad j=1,\cdots,n, \\
&\qquad \sum_{i=1}^{m} v_i x_{i\circ} = 1, \\
&\qquad u_r \geq \circ, \quad r=1,\cdots,s, \\
&\qquad v_i \geq \circ, \quad i=1,\cdots,m.
\end{aligned} \tag{8}$$

The CRR model in the output orientaion is as below:

$$\begin{aligned}
&\text{Max} \quad \varphi \\
&\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j X_j \leq X_\circ, \\
&\qquad \sum_{j=1}^{n} \lambda_j Y_j \geq \varphi Y_\circ, \\
&\qquad \lambda_j \geq \circ, \quad j=1,\cdots,n, \\
&\qquad v_i \geq \circ, \quad i=1,\cdots,m.
\end{aligned} \tag{9}$$

The above model is the CRR model in the output orientaion. This model is always feasible and φ^*=1. If φ^*=1, DMU0 is practical; otherwise, it is not. The two above models are called multiplication models with the output orientaion of CCR, which is as below:

$$\begin{aligned}
&\text{Max} \qquad V^t X_O, \\
&\text{s.t.} \\
&\qquad U^t Y_O = 1, \\
&\qquad V^t X_j - U^t Y_j \leq 0, \quad j=1,2,\ldots,n, \\
&\qquad U \geq 0, \ \ V \geq 0.
\end{aligned} \tag{10}$$

DMU0 is practical only and only if after solving the multiplication form of the CCR model in the optimal input orientaion (V^*, U^*), $U^{*t}Y_o$=1, (U^*, V^*).

By considering return technology to the changing scale of the BCC model, the input orientaion in the envelopment form is defined as below:

$$
\begin{aligned}
&\text{Min} \quad \theta, \\
&\text{s.t.} \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \quad i=1,2,\ldots,m, \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r=1,2,\ldots,s, \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1, \quad j=1,2,\ldots,n, \\
&\qquad \lambda_j \ge 0, \quad j=1,2,\ldots,n.
\end{aligned} \tag{11}
$$

This LP model is always practical and has a finite optimum, and always $0<\theta^{*}\le 1$.

The DMU of DMU0 in the BCC model is the efficiency of Pareto. If $\theta_B=1$, all subsidiary variables are zero in all optimal responses. The dual of this model is known as the multiplication form of BCC and is as below:

$$
\begin{aligned}
&\text{Max} \quad U^t Y_o + u_o, \\
&\text{s.t.} \\
&\qquad V^t X_o = 1, \\
&\qquad U^t Y_j - V^t X_j + u_o \le 0, \quad j=1,2,\ldots,n, \\
&\qquad U \ge 0, V \ge 0.
\end{aligned} \tag{12}
$$

2.2 | Network Data Envelopment Analysis

The network data envelopment analysis of DEA conventional models assumes DMUs to be a black box and neglects their internal structure. Färe [19], and Färe and Geraskove [20], [21] proposed network data envelopment analysis to overcome this problem as well as the problem of neglecting efficiency calculations. They believe that DEA conventional models overlook the organizational processes of the DMUs in their investigations and consider them as a black box, in which inputs are transformed into outputs without considering their internal structure. To improve performance, however, it is required to study different processes of the organization at different levels and divide successful parts from failed ones [22]. There are two common methods among the conventional DEA models to measure the efficiency of multiple parts organizations.

Accumulation (black box): As shown in *Fig. 1*, in a simple procedure, sections are accumulated and considered as a company. This procedure overlooks internal activities interaction and cannot calculate the impact of the inefficiency of sections on the entire efficiency of the company. In addition, this state can result in the improper selection of inputs, outputs, and non-logical evaluation of the DMU.

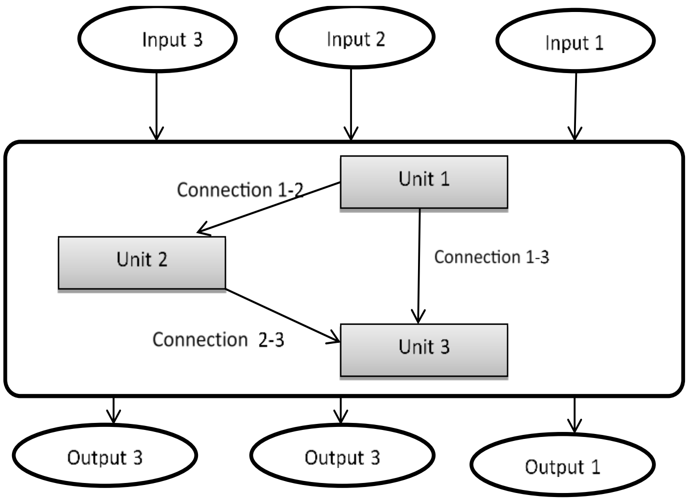

Fig. 1. Accumulation of the organization's units in the form of a black box.

Division: the second procedure involves measuring the efficiency of individual parts. Using this method, it is possible to evaluate the efficiency of each unit of the company among the DMUs. The procedure, however, is not practical for maintaining connectivity between units (see *Fig. 2*).

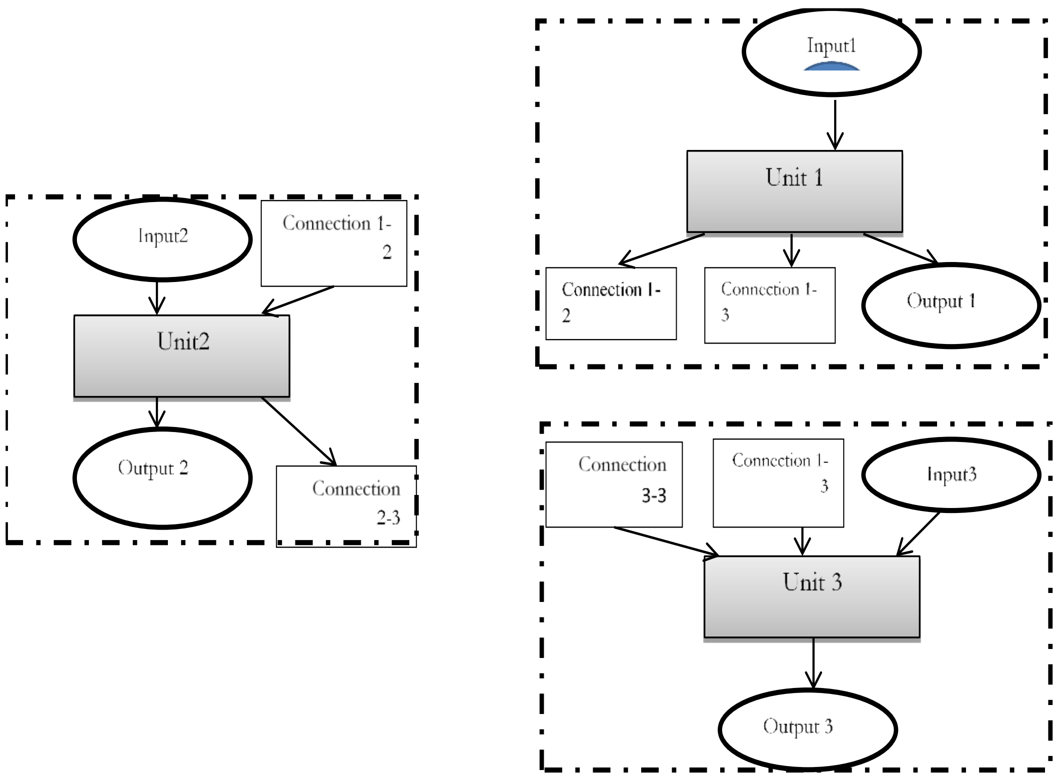

Fig. 2. Division of units of the organization.

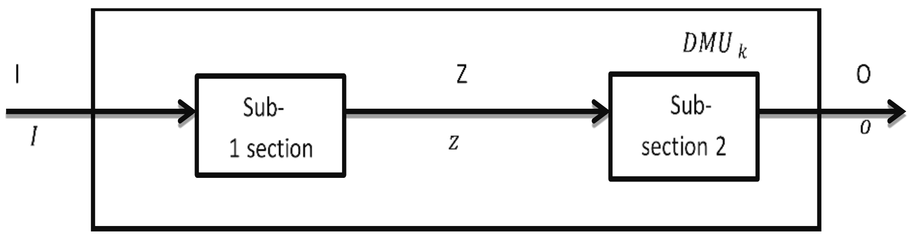

Fig. 3. Two-stage system.

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The overall form of the two-stage system is shown in *Fig. 3*, in which I, O, and Z are the input, the output of the DMUs, and the interconnection between sub-sections, respectively. The output of the first subsection is the input of the second subsection. The second subsection does not consume any exogenous input, and the first subsection does not create any exogenous output. Therefore, in 2008, the following model was presented for calculating the efficiency of a DMU with a two-stage structure, as shown in *Fig. 3* [12].

$$E_k^s = \max \sum_{r=1}^{s} u_r y_{rk} \Big/ \sum_{i=1}^{m} v_i x_{ik},$$

$$\text{s.t.}$$

$$\begin{aligned} &\sum_{r=1}^{s} u_r y_{rk} \Big/ \sum_{i=1}^{m} v_i x_{ik} \leq 1.\ \ j = 1. \dots . n, \\ &\sum_{p=1}^{q} w_p z_{pj} \Big/ \sum_{i=1}^{m} v_i x_{ij} \leq 1.\ \ j = 1. \dots . n, \\ &\sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{p=1}^{q} w_p z_{pj} \leq 1.\ \ j = 1. \dots . n, \\ &u_r . v_i . w_p \geq \varepsilon\ r = 1. \dots . s. i = 1. \dots . m. p = 1. \dots . q. \end{aligned} \tag{13}$$

2.3 | Supply Chain

In the global competition of the current era, various products should be provided regarding the customer's needs. The customer's desire for high quality and quick service has created pressure that has never been experienced before. In conclusion, companies cannot do everything alone. In the present competitive market, economic and production companies need management and monitoring resources and respective members outside the organization, in addition to considering internal resources. The reason is to achieve competitive advantages aiming to have a larger share of the market. Accordingly, activities such as supply and demand management, material provision, production management, good maintenance service, availability control, distribution, delivery, and service to the customer, which have already been noted in the country, were elevated to the supply chain level. The key factor in a supply chain is managing and controlling all these activities. Supply Chain Management (SCM) is a phenomenon to do this, and customers can receive reliable and fast service with a high quality and low price. In the 1960-70 decades, organizations paid more attention to developing market strategies focused on satisfying customers. They found that strong engineering, design, and harmonic production operation are required to achieve market demands and, consequently larger share of the market. It is thus imperative that designers incorporate the ideals and requirements of customers when designing productions and present them to the market at a minimum price while maintaining the maximum possible level of quality. In 1990, along with improving production abilities, industrial managers found that receiving materials and services from different providers significantly affected increasing the abilities of organizations to meet customer requirements. It influences the organization's focus, supply bases, and resource-finding strategies. Managers also found that merely producing a production is not enough. In fact, the provision of products with criteria of the customers (when, where, how) and their required cost and quality created new challenges. In this circumstance, they found from the above changes that these changes are not sufficient in the long-term to manage their organization. They should have been involved in the network management of all factories and companies that provided the input of their organization directly and indirectly, and in companies related to delivery and after-sale services. Regarding this vision, supply chain procedures and management appeared.

Therefore, one can say that the supply chain includes all stages which directly and indirectly affect meeting the requirements of a customer. In an ordinary supply chain, raw materials are sent from providers to factories. Then, products are delivered to central and distributor warehouses to get to the final customers or consumers. Then, the good passes through different steps of a chain to get to the consumer. In some of these stages, the good is stored, and in others, it is transported. It means the supply chain is a set of storage and transportation. Members of an ordinary supply chain are providers, ingredient warehouses, production centers, distributors, retailers, and final customers. Each commercial organization is a unit of the supply chain, and many organizations are units of several supply chains. A supply chain's number and type are determined by specifying which organization is the producer or beneficiary. Traditionally, a supply chain consists of the below stages or cycles (see *Fig. 4*).

Fig. 4. Supply chain.

3 | Performance Evaluation of the Supply Chain of Petrochemical Using Data Envelopment Analysis

Fig. 5 shows a three-stage chain (network). As we will discuss in the next section, this structure is derived from the petrochemical chain. Despite being part of the petrochemical chain, we know of many petrochemical units that share similar production characteristics, so we can compare them. In this figure, the first stage has an independent input of X1. Also, a returned output from the second stage, Y, will enter this stage, exit from the first stage of the Z output, which is a mediator production, and enters the second stage. The first stage has another output, D, which exit as the final output. The mediator production of Z is the input of the second stage. Additionally, this stage has another input stage, X2, which enters from outside the system. Y represents the mediator stage of the second stage of production. Mediator productions of the second stage return to the first stage, and another proceeds to the third stage. The third stage has three other independent inputs, X3, which enters the system from outside. Finally, an output, D, exits the third stage as the final output.

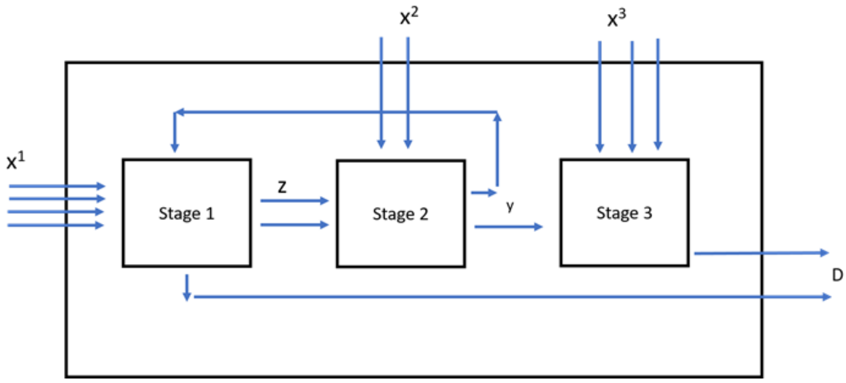

Fig. 5. Three-stage network (chain) based on the petrochemical process structure.

Analysis and investigation of network units in the data envelopment analysis attracted attention after presenting the DEA classic models, which considered DMUs as a black box. In this procedure, the impact of relationships between stages and considering this equation in the numerical modeling for evaluation is required. Because considering the internal structure of a DMU, composed of different components, affects the network evaluation considerably. This research involved modeling and evaluation, as shown in *Fig. 1*. We examine modeling for efficiency evaluation, finding the pattern, and obtaining improved activities for the entire network and each stage. We follow this numerical modeling in DEA with both envelopment and multiplication forms, and we try to consider different states of modeling to achieve overall efficiency, aggregate efficiency, and stage efficiency.

3.1 | Modeling in the Envelopment form of the Data Envelopment Analysis

I. Envelopment form model of the input orientaion

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Consider the input orientaion envelopment form of the above network. In this model, efficiency has been minimized in terms of inputs. Note that all external inputs that enter the three-stages of thenetwork were minimized.

$$
\begin{aligned}
& \text{Min} \quad \theta, \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 \le \theta x_{io}^1, \qquad i = 1,\dots,4, \\
& \sum_{j=1}^{n} \lambda_j^1 d_{1j} \ge d_{1o}, \\
& \sum_{j=1}^{n} \lambda_j^1 z_{fj} \ge z_{fo}, \qquad f = 1,2, \\
& \sum_{j=1}^{n} \lambda_j^1 y_{1j} \ge y_{1o}, \\
& \sum_{j=1}^{n} \lambda_j^2 x_{lj}^2 \le \theta x_{lo}^2, \qquad l = 1,2, \\
& \sum_{j=1}^{n} \lambda_j^2 z_{fj} \le z_{fo}, \qquad f = 1,2, \\
& \sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}, \qquad r = 1,2, \\
& \sum_{j=1}^{n} \lambda_j^3 x_{pj}^3 \le \theta x_{po}^3, \qquad l = 1,2,3, \\
& \sum_{j=1}^{n} \lambda_j^3 d_{2j} \le d_{2o}, \\
& \sum_{j=1}^{n} \lambda_j^3 y_{2j}^3 \ge y_{2o}, \\
& \sum_{j=1}^{n} (\lambda_j^s) = 1, \qquad s=1,2,3, \\
& \lambda_j^s \ge 0, \qquad j = 1,\dots,n, s = 1,2,3.
\end{aligned} \tag{14}
$$

This model is used to determine efficiency and improved inputs. If we accept that θ^* is the value of the overall efficiency of the supply chain system, the corresponding formula of the improved inputs is:

$$\text{improved inputs} = (\theta^* x_o^1,\ \theta^* x_o^2,\ \theta^* x_o^3).$$

We prove two below theorems to analyze the noted model.

Theorem 1. From *Model (1)*, $\theta^* \le 1$.

Theorem 2. Improved inputs in the DMUo evaluation unit, which is calculated from $(\theta^* x_o^1, \theta^* x_o^2, \theta^* x_o^3)$, are on the weak efficiency border.

Proving Theorem 2. From *Model (14)*, $\theta^* \leq 1$. Since there is a practical $\theta = 1$, answer for *Model (15)*, and this model is from the minimization type, then $\theta^* \leq 1$.

Theorem 3. Improved inputs in the DMUo unit are calculated from $(\theta^* x_o^1, \theta^* x_o^2, \theta^* x_o^3)$, and are in the weak efficiency border.

Proving Theorem 3. By absurd hypothesis, the point is not on the weak efficiency border, so a smaller value than θ^*, such as $\overline{\theta}^*$,$\theta^* < \theta^*$can be found, to $\left(\theta^* x_o^1, \theta^* x_o^2, \theta^* x_o^3\right)$ be on the weak efficiency border, which is against the assumption of θ^* being optimum. Then the absurd hypothesis is wrong, and the verdict is valid.

Consider the output orientaion of the envelopment form corresponding to the above network. In this model, the efficiency was maximized in terms of outputs. Note that all outputs quitting the triple stages of the network were maximized.

II. Envelopment form model of the output orientaion.

Consider the output orientaion of the envelopment form corresponding to the above network. In this model, the efficiency was maximized in terms of outputs. Note that all external outputs entering the triple stages of the network were maximized.

$$
\begin{aligned}
&\text{Max} \quad \varphi \\
&\text{s.t.} \\
&\quad \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 \leq x_{io}^1, && i = 1,\dots,4, \\
&\quad \sum_{j=1}^{n} \lambda_j^1 d_{1j} \geq \varphi d_{1o}, \\
&\quad \sum_{j=1}^{n} \lambda_j^1 z_{fj} \geq z_{fo}, && f = 1,2, \\
&\quad \sum_{j=1}^{n} \lambda_j^1 y_{1j} \geq y_{1o}, \\
&\quad \sum_{j=1}^{n} \lambda_j^2 x_{lj}^2 \leq x_{lo}^2, && l = 1,2, \\
&\quad \sum_{j=1}^{n} \lambda_j^2 z_{fj} \leq z_{fo}, && f = 1,2, \\
&\quad \sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}, && r = 1,2, \\
&\quad \sum_{j=1}^{n} \lambda_j^3 x_{pj}^3 \leq x_{po}^3, && l = 1,2,3, \\
&\quad \sum_{j=1}^{n} \lambda_j^3 d_{2j} \leq \varphi d_{2o},
\end{aligned}
\tag{15}
$$

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$$\sum_{j=1}^{n}\lambda_j^3 y_{2j}^3 \geq y_{2o},$$

$$\sum_{j=1}^{n}(\lambda_j^s) = 1, \qquad s=1,2,3,$$

$$\lambda_j^s \geq 0, \qquad j=1,\dots,n, s=1,2,3.$$

This model is used to determine efficiency and improved inputs. The corresponding formula of the improved inputs is:

$$\text{Improved Outputs} = (\varphi^* d_o).$$

We prove two below theorems to analyze the noted model:

Theorem 4. From *Model (15)*, $\varphi^* \geq 1$.

Theorem 5. Improved inputs in the DMUo evaluation unit, which is calculated from ($\varphi^* d_o$), are on the weak efficiency border.

Theorem 6. From *Model (15)*, $\varphi^* \geq 1$.

Proving Theorem 6. From *Model (14)*, $\theta^* \leq 1$. Since there is a practical $\varphi = 1$ answer for *Model (15)*, and this model is from the maximizing type, then $\varphi^* \geq 1$.

Theorem 7. Improved inputs in the DMUo unit are calculated from ($\varphi^* d_o$), and are in the weak efficiency border.

Proving Theorem 7. By absurd hypothesis, the point is not on the weak efficiency border, so a larger value than φ^* , such as $\overline{\varphi}^*$, φ^* can be found, to ($\varphi^* d_o$) be on the weak efficiency border, which is against the assumption of φ^* being optimum. Then the absurd hypothesis is wrong, and the verdict is valid.

An index can be introduced considering *Models (15)* and *(16)* by combining the optimum responses of these models as below. If the input efficiency of the chain axis is θ^* and the output efficiency of its axis is φ^*, the combined equation is as below:

$$\text{Efficiency Index (E.I)} = \frac{1}{w_1\theta^* + w_2\varphi^*},$$

where $w_1, w_2 \in R^+, w_1 + w_2 = 1$.

In other words, w1 and w2 weights are determined by the system manager or expert and represent the importance of $\theta^*$and φ^*concerning each other. It is evident that $0 \prec E.\,I \leq 1$.

The supply chain efficiency is within the range of 0 and 1. According to the envelopment analysis fundamentals, if the efficiency is 1, then the structure is efficient. To do this, the below theorem can be proved.

Theorem 8. If E. I=1 in the DMUo evaluation, the unit is efficient, and if E. I<1, the evaluated unit is inefficient.

Proving Theorem 8. If E. I=1, we have $w_1\theta^* + w_2\varphi^* = 1$. since $w_1, w_2 \in R^+, w_1 + w_2 = 1$ and $\theta^* \leq 1, \varphi^* \geq 1$.

It was found that $\theta^* = \varphi^* = 1$. Then the efficiency unit is efficient. Because there is no suggested change in the input or outputs of the model.

If $E.I < 1$, then $w_1\theta^* + w_2\varphi^* > 1$, since $w_1, w_2 \in R^+, w_1 + w_2 = 1$, it was concluded that $\varphi^* > 1$ or $\theta^* < 1$. If one or both noted equations are validated, we conclude that the evaluation unit can reduce its inputs or increase its outputs. Since there is a dominant unit for it, it is inefficient.

3.2 | Modeling in the Envelopment form of the Data Envelopment Analysis

We consider the noted network (1) to write the multiplication form model in two input and output orientaions.

I. Multiplication form model in the input orientation.

In this model, the corresponding optimum weight of the input and outputs of each stage is determined, in addition to the efficiency. Note that the stage efficiency should be equal to or lower than 1.

$$
\begin{aligned}
&\text{Max} \ \sum_{r=1}^{2} k_r d_{ro}, \\
&\text{s.t.} \\
&\quad \sum_{i=1}^{3} v_i^1 x_{io}^1 + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3 = 1, \\
&\quad \sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j} - \sum_{i=1}^{3} v_i^1 x_{ij}^1 - k_1 y_{1j} + q_0^1 \le 0, \qquad j=1,\dots,n, \\
&\quad \sum_{r=1}^{2} k_r y_{rj} - \sum_{l=1}^{2} v_l^2 x_{lj}^2 - \sum_{f=1}^{2} w_f z_{fj} + q_0^2 \le 0, \qquad j=1,\dots,n, \\
&\quad u_2 d_{2j} - \sum_{p=1}^{3} v_p^3 x_{pj}^3 - k_2 y_{2j} + q_0^3 \le 0, \qquad j=1,\dots,n, \\
&\quad \sum_{r=1}^{2} k_r d_{rj} - \sum_{i=1}^{3} v_i^1 x_{ij}^1 - \sum_{l=1}^{2} v_l^2 x_{lj}^2 - \sum_{p=1}^{3} v_p^3 x_{pj}^3 \le 0, \qquad j=1,\dots,n, \\
&\quad k_r \ge 0, v_i^1 \ge 0, v_l^2 \ge 0, v_p^3 \ge 0, w_f \ge 0, u_s \ge 0 \quad \text{for all} \ \ r,i,l,p,f,s, \\
&\quad q_0^1,\ q_0^2,\ q_0^3 \ \ \text{Free.}
\end{aligned}
\tag{16}
$$

Theorem 9. It is proved that $\sum_{r=1}^{2} k_r^* d_{ro} \le 1$.

Proving Theorem 9. Considering that *Model (16)* is the dual of *Model (14)*, the finite optimum response of these two problems has a similar value of the goal function. Therefore: $\sum_{r=1}^{2} k_r^* d_{ro} \le 1$.

II. Multiplication form model in the output orientation.

We write the Multiplication form model in the input orientaion for *Fig. 1* to be in accordance with the *Eq. (4)*. In this model, the corresponding optimum weight of the input and outputs of each stage is determined, in addition to the efficiency. Note that the stage efficiency should be equal to or larger than 1.

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$$\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{3} v_i^1 x_{io}^1 + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3 \\
\text{s.t.} \quad & \\
& \sum_{r=1}^{2} k_r d_{ro} = 1, \\
& \sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} - \sum_{f=1}^{2} w_f z_{fj} - u_1 d_{1j} + t_0^1 \geq 0, \qquad j=1,\dots,n, \\
& \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{f=1}^{2} w_f z_{fj} - \sum_{r=1}^{2} k_r y_{rj} + t_0^2 \geq 0, \qquad j=1,\dots,n, \\
& \sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j} - u_2 d_{2j} + t_0^3 \geq 0, \qquad j=1,\dots,n, \\
& \sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3 - \sum_{r=1}^{2} k_r d_{rj} \geq 0, \qquad j=1,\dots,n, \\
& k_r \geq 0, v_i^1 \geq 0, v_l^2 \geq 0, v_p^3 \geq 0, w_f \geq 0, u_s \geq 0 \quad \text{for all } r,i,l,p,f,s, \\
& t_0^1, t_0^2, t_0^2 \ \text{Free.}
\end{aligned} \tag{17}$$

Theorem 10. It is proved that:

$$\sum_{i=1}^{3} v_i^{1*} x_{io}^1 + \sum_{l=1}^{2} v_l^{2*} x_{lo}^2 + \sum_{p=1}^{3} v_p^{3*} x_{po}^3 \geq 1.$$

Proving Theorem 10. By considering that *Model (17)* is the dual of *Model (15)*, the finite optimum response of these two problems has a goal function equal to $\sum_{i=1}^{3} v_i^{1*} x_{io}^1 + \sum_{l=1}^{2} v_l^{2*} x_{lo}^2 + \sum_{p=1}^{3} v_p^{3*} x_{po}^3 \geq 1$. Therefore, one can change modeling in *Models (16)* to obtain the stage efficiency. The mathematical equations are shown below to calculate the efficiency of various stages. If ei is the value of the efficiency of different stages of this supply chain, we can calculate the values of these efficiencies according to the below equations:

$$\text{Efficiency Stage } 1 = e_1 = \frac{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j} + q_0^1}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j}} \leq 1, \qquad j=1,\dots,n,$$

$$\text{Efficiency Stage } 2 = e_2 = \frac{\sum_{r=1}^{2} k_r y_{rj} - \sum_{l=1}^{2} v_l^2 x_{lj}^2 + q_0^1}{\sum_{f=1}^{2} w_f z_{fj}} \leq 1, \qquad j=1,\dots,n,$$

$$\text{Efficiency Stage } 3 = e_3 = \frac{u_2 d_{2j} + q_0^3}{\sum_{p=1}^{3} v_p^3 x_{pj}^3 - k_2 y_{2j}} \leq 1, \qquad j=1,\dots,n,$$

$$\text{Efficiency Overall} = e_o = \frac{\sum_{r=1}^{2} u_r d_{rj}}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3} \leq 1, \qquad j=1,\dots,n.$$

Consider *Model (17)*. An important principle of this model is to determine if the input efficiency of the entire supply chain can be obtained based on input reduction. If we change *Model (16)* as described below, we can interpret the efficiency of the stage based on the model. Below is a formula that describes the stage efficiency and the entire network following the presented model.

Notably, a crucial equation in calculating the entire system efficiency is using an aggregate relationship. The efficiency of the first, second, and third stages are e1, e2, and e3, respectively, and by considering the aggregate equation, the overall efficiency is defined as below:

$$\text{Efficiency Aggregate } = e^a = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3.$$

Substitution of their equivalent defined mathematical equations gives us the aggregate efficiency as below:

$$\text{Efficiency Aggregate} = e^a = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3,$$

$$e^a = \mu_1 \frac{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j} + q_0^1}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j}} + \mu_2 \frac{\sum_{r=1}^{2} k_r y_{rj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + q_0^2}{\sum_{f=1}^{2} w_f z_{fj}} + \mu_3 \frac{u_2 d_{2j} + q_0^3}{\sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j}} \quad j=1,\dots,n.$$

Values of μ_i show the efficiency value weights of each stage. We define weights as below. Each fraction for each DMUj is the ratio of the utilized inputs in each stage to independent inputs (independent inputs enter each stage outside the system) of the system.

$$\mu_1 = \frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j}}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3} \, j=1,\dots,n,$$

$$\mu_2 = \frac{\sum_{f=1}^{2} w_f z_{fj}}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3} \, j=1,\dots,n,$$

$$\mu_3 = \frac{\sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j}}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3} = 1,\dots,n.$$

Therefore, the simple form of the defined state for the introduced network system aggregate efficiency in *Fig. 1* is as below:

$$e^a = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 = \frac{\sum_{f=1}^{2} w_f z_{fo} + u_1 d_{1o} + u_2 d_{2o} + \sum_{r=1}^{2} k_r y_{rj}}{\sum_{i=1}^{3} v_i^1 x_{io}^1 + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3}, \quad j=1,\dots,n.$$

We overlooked the free variable in the numerator of the fraction after simplifying the introduced state in the equation above. *Model (18)* is derived from the above equation. As shown in the following model, the goal is to maximize the aggregate efficiency of the network if both the overall efficiency and the stage efficiency are less than 1. See the following model:

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$$\text{Max}\quad \frac{\sum_{f=1}^{2} w_f z_{fo} + u_1 d_{1o} + u_2 d_{2o} + \sum_{r=1}^{2} k_r y_{ro}}{\sum_{i=1}^{3} v_i^1 x_{io}^1 + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3},$$

s.t.

$$\frac{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j} + q_0^1}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j}} \le 1, \qquad j=1,\ldots,n,$$

$$\frac{\sum_{r=1}^{2} k_r y_{rj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + q_0^2}{\sum_{f=1}^{2} w_f z_{fj}} \le 1, \qquad j=1,\ldots,n,$$

$$\frac{u_2 d_{2j} + q_0^3}{\sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j}} \le 1, \qquad j=1,\ldots,n, \tag{18}$$

$$\frac{\sum_{r=1}^{2} u_r d_{rj}}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3} \le 1, \qquad j=1,\ldots,n,$$

$$\frac{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j} + u_2 d_{2j} + \sum_{r=1}^{2} k_r y_{rj}}{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3} \le 1, \qquad j=1,\ldots,n,$$

$$k_r \ge 0, v_i^1 \ge 0, v_l^2 \ge 0, v_p^3 \ge 0, w_f \ge 0, u_s \ge 0 \quad \text{for all } r,i,l,p,f,s,$$

$$q_0^1,\ q_0^2,\ q_0^3 \ \text{Free.}$$

After the application of the variable transformation, $h = \frac{1}{\sum_{i=1}^{3} v_i^1 x_{io}^1 + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3}$ we transform the above model, which has a nonlinear goal function, to a linear problem. So, by applying the above variable transformation we will have:

$$hk_r = k_r, hv_i^1 = v_i^1, hv_l^2 = v_l^2, hv_p^3 = v_p^3, hw_f = w_f, hu_s = u_s \quad \text{for all } r,i,l,p,f,s$$
$$hq_0^1 = q_0^1,\ hq_0^2 = q_0^2,\ hq_0^3 = q_0^3, hq = q.$$

To simplify formulation, we used the previous variable name after the variable transformation.

$$hk_r = k_r, hv_i^1 = v_i^1, hv_l^2 = v_l^2, hv_p^3 = v_p^3, hw_f = w_f, hu_s = u_s \quad \text{for all } r,i,l,p,f,s$$
$$hq_0^1 = q_0^1,\ hq_0^2 = q_0^2,\ hq_0^3 = q_0^3, hq = q.$$

Therefore, considering h>0, after applying variable transformation of all nonnegative variables, they remain nonnegative. On the other hand, all free variables in the sign remain free in the sign. After variable transformation, we will have:

$$\begin{aligned}
\text{Max}\quad & \sum_{f=1}^{2} w_f z_{fo} + u_1 d_{1o} + u_2 d_{2o} + \sum_{r=1}^{2} k_r y_{ro}, \\
\text{s.t.}\quad & \\
& \sum_{i=1}^{3} v_i^1 x_{io}^1 + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3 = 1, \\
& \sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j} - \sum_{i=1}^{3} v_i^1 x_{ij}^1 - k_1 y_{1j} + q_0^1 \le 0, \qquad j=1,\dots,n, \\
& \sum_{r=1}^{2} k_r y_{rj} - \sum_{l=1}^{2} v_l^2 x_{lj}^2 - \sum_{f=1}^{2} w_f z_{fj} + q_0^2 \le 0, \qquad j=1,\dots,n, \\
& u_2 d_{2j} - \sum_{p=1}^{3} v_p^3 x_{pj}^3 - k_2 y_{2j} + q_0^3 \le 0, \qquad j=1,\dots,n, \\
& \sum_{r=1}^{2} u_r d_{rj} - \sum_{i=1}^{3} v_i^1 x_{ij}^1 - \sum_{l=1}^{2} v_l^2 x_{lj}^2 - \sum_{p=1}^{3} v_p^3 x_{pj}^3 \le 0, \qquad j=1,\dots,n, \\
& \sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j} + u_2 d_{2j} + \sum_{r=1}^{2} k_r y_{rj} - \sum_{i=1}^{3} v_i^1 x_{ij}^1 - \sum_{l=1}^{2} v_l^2 x_{lj}^2 - \sum_{p=1}^{3} v_p^3 x_{pj}^3 \le 0, \qquad j=1,\dots,n, \\
& k_r \ge 0, v_i^1 \ge 0, v_l^2 \ge 0, v_p^3 \ge 0, w_f \ge 0, u_s \ge 0 \quad \text{for all } r,i,l,p,f,s, \\
& q_0^1,\ q_0^2,\ q_0^3 \ \text{Free.}
\end{aligned} \tag{19}$$

It is possible to calculate the overall efficiency, the stage efficiency, and the aggregate efficiency based on the optimum response of the above model. All these values were formulated from changing inputs. In other words, models are in the input orientation.

The stage efficiency can be calculated based on the output orientaion. The *Model (19)* can be modified to achieve efficiency at each stage. Consider the *Model (19)*. This model is based on an evaluation of the efficiency of the output of the entire supply chain if the output efficiency has been achieved. We assume that bi is the efficiency value of each stage. We can interpret the efficiency of stages from the model by using the below equations. Following is a formula for determining the stage efficiency and the overall efficiency of the network based on the presented model.

$$\text{Efficiency Stage } 1 = b_1 = \frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} + t_0^1}{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j}} \ge 1, \qquad j=1,\dots,n,$$

$$\text{Efficiency Stage } 2 = b_2 = \frac{\sum_{f=1}^{2} w_f z_{fj} + t_0^2}{\sum_{r=1}^{2} k_r y_{rj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2} \ge 1, \qquad j=1,\dots,n,$$

$$\text{Efficiency Stage } 3 = b_3 = \frac{\sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j} + t_0^3}{u_2 d_{2j}} \ge 1, \qquad j=1,\dots,n,$$

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$$\text{Efficiency Overall} = b_o = \frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3}{\sum_{r=1}^{2} u_r d_{rj}} \geq 1, \qquad j=1,\dots,n.$$

As previously discussed, the aggregate efficiency can be defined as follows. The aggregate efficiency is composed of the summation of the weighted efficiency of stages.

$$\text{Efficiency Aggregate} = b^a = \delta_1 b_1 + \delta_2 b_2 + \delta_3 b_3,$$

$$b^a = \delta_1 \frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} + t_0^1}{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j}} + \delta_2 \frac{\sum_{f=1}^{2} w_f z_{fj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + t_0^2}{\sum_{r=1}^{2} k_r y_{rj}} + \delta_3 \frac{\sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j} + t_0^3}{u_2 d_{2j}}, \quad j=1,\dots,n\,.$$

Values of μi show the efficiency value weights of each stage. We define weights as below. Each fraction for each DMUj is the ratio of the produced outputs in each stage to independent outputs (independent outputs exit as the final product from the entire system) in the entire system.

$$\mu_1 = \frac{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j}}{u_1 d_{1j} + u_2 d_{2j}} \quad j=1,\dots,n,$$

$$\mu_2 = \frac{\sum_{r=1}^{2} k_r y_{rj}}{u_1 d_{1j} + u_2 d_{2j}} \quad j=1,\dots,n,$$

$$\mu_3 = \frac{u_2 d_{2j}}{u_1 d_{1j} + u_2 d_{2j}} \quad j=1,\dots,n.$$

Therefore, the simple form of the defined state for the introduced network system aggregate efficiency in *Fig. 1* is as below:

$$b^a = \delta_1 b_1 + \delta_2 b_2 + \delta_3 b_3 = \frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} + \sum_{f=1}^{2} w_f z_{fj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3 + k_2 y_{2j}}{u_1 d_{1j} + u_2 d_{2j}} \quad j=1,\dots,n.$$

We know that in the above equation, the goal is to minimize the aggregate efficiency of the network if both the overall efficiency and the stage efficiency are less than 1. See the following model:

$$\text{Min} \quad \frac{\sum_{i=1}^{3} v_i^1 x_{io}^1 + k_1 y_{1o} + \sum_{f=1}^{2} w_f z_{fo} + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3 + k_2 y_{2o}}{u_1 d_{1o} + u_2 d_{2o}},$$

s.t.

$$\frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} + t_0^1}{\sum_{f=1}^{2} w_f z_{fj} + u_1 d_{1j}} \geq 1, \qquad j=1,\dots,n,$$

$$\frac{\sum_{f=1}^{2} w_f z_{fj} + t_0^2}{\sum_{r=1}^{2} k_r y_{rj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2} \geq 1, \qquad j=1,\dots,n,$$

$$\frac{\sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j} + t_0^3}{u_2 d_{2j}} \geq 1, \qquad j=1,\dots,n,$$

$$\frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3}{\sum_{r=1}^{2} u_r d_{rj}} \geq 1, \qquad j=1,\dots,n,$$

$$\frac{\sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} + \sum_{f=1}^{2} w_f z_{fj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j}}{u_1 d_{1j} + u_2 d_{2j}} \geq 1 \quad j=1,\dots,n,$$

$$k_r \geq 0, v_i^1 \geq 0, v_l^2 \geq 0, v_p^3 \geq 0, w_f \geq 0, u_s \geq 0, \quad \text{for all } r,i,l,p,f,s,$$
$$t_0^1,\ t_0^2,\ t_0^3 \ \text{Free.}$$

We overlooked the free variable in the numerator of the fraction, which was created after simplifying the introduced state in the equation above.

After the application of the variable transformation, $\frac{1}{u_1 d_{1j} + u_2 d_{2j}} = c$, we transform the above model, which has a nonlinear goal function, to a linear problem. So, by applying the above variable transformation we will have:

$$ck_r = k_r, cv_i^1 = v_i^1, cv_l^2 = v_l^2, cv_p^3 = v_p^3, cw_f = w_f, cu_s = u_s \quad \text{for all } r,i,l,p,f,s.$$
$$ct_0^1 = t_0^1,\ ct_0^2 = t_0^2,\ ct_0^3 = t_0^3,\ ct = t.$$

To simplify formulation, we used the previous variable name after the variable transformation.

To simplify formulation, we used the previous variable name after the variable transformation. Therefore, considering c>0, after applying variable transformation of all nonnegative variables, they remain nonnegative. On the other hand, all free variables in the sign remain free in the sign. After variable transformation, we will have:

$$k_r \geq 0, v_i^1 \geq 0, v_l^2 \geq 0, v_p^3 \geq 0, w_f \geq 0, u_s \geq 0 \quad \text{for all } r,i,l,p,f,s,$$
$$t_0^1,\ t_0^2,\ t_0^3,\ t \ \text{Free.}$$

Therefore, the below model is driven after simplification:

$$\begin{aligned}
&\text{Min} \quad \sum_{i=1}^{3} v_i^1 x_{io}^1 + k_1 y_{1o} + \sum_{f=1}^{2} w_f z_{fo} + \sum_{l=1}^{2} v_l^2 x_{lo}^2 + \sum_{p=1}^{3} v_p^3 x_{po}^3 + k_2 y_{2o} \\
&\text{s.t.} \\
&\quad u_1 d_{1o} + u_2 d_{2o} = 1, \\
&\quad \sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} - \sum_{f=1}^{2} w_f z_{fj} - u_1 d_{1j} + t_0^1 \geq 0, \quad j=1,\dots,n, \\
&\quad \sum_{f=1}^{2} w_f z_{fj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2 - \sum_{r=1}^{2} k_r y_{rj} + t_0^2 \geq 0, \quad j=1,\dots,n, \\
&\quad \sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j} - u_2 d_{2j} + t_0^3 \geq 0, \quad j=1,\dots,n, \qquad (20) \\
&\quad \sum_{i=1}^{3} v_i^1 x_{ij}^1 + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3 - \sum_{r=1}^{2} u_r d_{rj} \geq 0, \quad j=1,\dots,n, \\
&\quad \sum_{i=1}^{3} v_i^1 x_{ij}^1 + k_1 y_{1j} + \sum_{f=1}^{2} w_f z_{fj} + \sum_{l=1}^{2} v_l^2 x_{lj}^2 + \sum_{p=1}^{3} v_p^3 x_{pj}^3 + k_2 y_{2j} - u_1 d_{1j} + u_2 d_{2j} \geq 0 \quad j=1,\dots,n, \\
&\quad k_r \geq 0, v_i^1 \geq 0, v_l^2 \geq 0, v_p^3 \geq 0, w_f \geq 0, u_s \geq 0 \quad \text{for all } r,i,l,p,f,s, \\
&\quad t_0^1,\ t_0^2,\ t_0^3 \ \text{Free.}
\end{aligned}$$

From the optimal response of the above model, one can calculate overall efficiency, the stage efficiency, and aggregate efficiency. All these values were formulated based on changing outputs. In other words, models are in the output orientaion. In Section 4, the above models were implemented.

4 | A Practical Example in the Petrochemical Industry

Many of the DMUs have more than one stage. By evaluating the performance of these units using data envelopment analysis, the entire DMU cannot be viewed as a black box. Rather, the internal equations should also be taken into consideration. Different methods were presented to study the efficiency of multistage units.

As noted in Section 3, two different methods were introduced considering the multiplication model to investigate a supply chain, including mediator and reversible relationships along with independent inputs and outputs of the system. One of these methods focused on investigating the aggregate efficiency of the chain, and the other one aimed to evaluate the overall efficiency of the system in terms of input and output orientaion. We evaluated the efficiency of the network stages in both states. Each of the methods has its unique theoretical properties.

In this section, we examine a practical example in the petrochemical industry and implement both introduced procedures in this practical example to analyze the obtained results. A part of the petrochemical industry has been extracted. In addition, we will examine the aggregate and overall efficiency in terms of input and output orientaion, considering the proposed model. Since our objective was to evaluate the results of input and output orientaion, we did not use the hybrid nature of data envelopment analysis.

4.1 | Different Forms of Implementing Decision-Making Units

A DMU transforms data into outputs. In the DEA model, DMUs should be homogenous and have similar tasks and goals. This method measures efficiency by considering the ratio of different inputs (or resources) to different produced outputs (services). Therefore, the variables of the problem can be divided into two overall groups of data and outputs. Determining data and output variables is crucial in implementing the DEA model because the results of this model are based on the selected data and outputs, and changing data or output will change the model results. Therefore, a correct definition of data and output variable gives a realistic efficiency of DMUs. The extraction of evaluation inputs and outputs, which were selected among a set of indices, is the most critical part of the research. It should be noted that considering different goals in evaluation results in selecting various input and output indices. However, the role of indices is to warn decision-makers about potential or hidden problems in specific fields or to continue the desired process in other fields.

The overall process of different petrochemicals was examined to identify indices. Then, experts verified the process of the below figure (see *Fig. 6*).

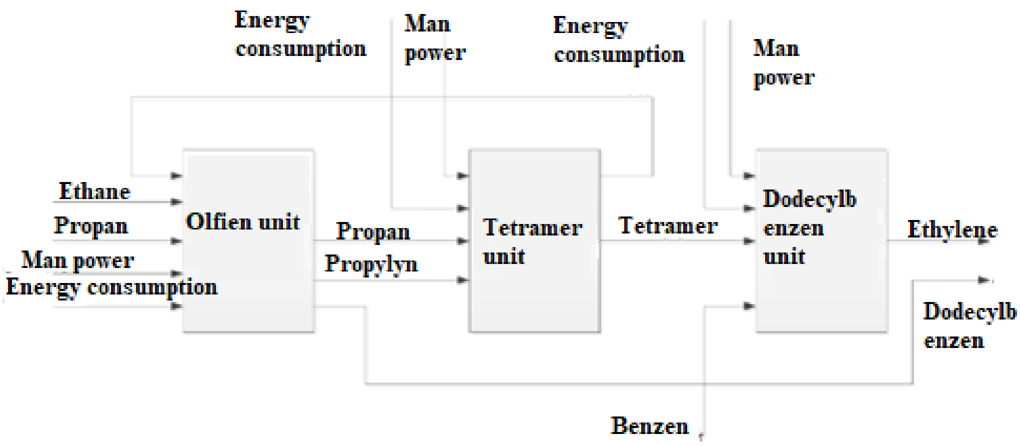

Fig. 6. Petrochemical process.

The figure shows processes in three stages of olefin, tetramer, and dodecylbenzene units. Each unit has specific tasks that are described below:

Industries that transform hydrocarbon of crude oil or natural gas is transformed to a new chemical substance called petrochemical. In some cases, in petrochemical production, a principal upper unit produces raw material for other units. For example, the olefin unit provides the demand for polyethylene and polypropylene units by creating ethylene and propylene. Therefore, the energy state in each unit is examined separately, considering the process difference and diversity.

However, in this industry, like the refinery industries, some units, such as olefine, consume fuel as feed. Consumable energy carriers in petrochemical complexes are often natural gas and fossil fuel. In Petrochemical production, a principal upper unit can produce raw materials for other units. For example, the olefin unit provides the demand for polyethylene, tetramer, and propyl units by creating ethylene and propyl.

After several investigations, indices were completed corresponding to the chain structure. Consultations were made with petrochemicals that have similar processes to extract data from 20 petrochemical units. *Tables 1* and *2* show indices:

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Table 1. Supply chain data in the first stage.

DMUs	Stage 1 Inputs x11 Ethane Value (Thousad Tons)	x12 Propane Value (Thousad Tons)	x13 Man Power Number	x14 Unit Cost	x15 Returned Propane (Thousand Tons)	Intermediate Z11 Propane Value (Thousad Tons)	Z12 Propylene Amount	Output d Ethylene Cost (in Thousands of Dollars)
1	620	325	64	4480	82	117	387	333650
2	540	280	73	5110	75	120	260	322500
3	720	460	123	8610	200	240	530	307500
4	120	1000	38	2660	122	147	650	81000
5	650	390	96	6720	135	170	290	43500
6	750	450	84	5880	180	260	370	427500
7	910	650	101	7070	215	370	440	562500
8	720	480	84	5880	190	320	280	450000
9	420	225	34	3480	132	217	387	433650
10	340	580	83	6110	95	320	460	622500
11	340	360	73	1610	100	140	130	207500
12	160	900	154	8660	202	127	750	61000
13	450	190	56	8720	235	270	390	83500
14	820	850	74	2880	120	160	870	327500
15	820	350	91	2070	195	470	940	862500
16	900	880	64	8880	290	120	580	750000
17	270	625	104	7480	172	470	790	733650
18	820	580	118	9110	175	460	270	522500
19	470	825	94	6480	182	570	940	807500
20	520	280	88	6110	145	120	880	765490

Table 2. Supply chain data in the second and third stages.

DMUs	Stage 2 Input x2 Man Power Tetramer Unit	x2 Unit Cost	Intermediate y Tetramer Value (Thousad Tons)	y	Stage 3 Inputs x3 Man Power Number	x3 Dodecylbenzene (Thousand Tons)	x3 Unit Cost	Output d Benzene Value (Thousand Tons)
1	47	3456	317	182	42	447	4536	143
2	53	3786	360	175	35	472	6543	120
3	68	4327	550	200	35	620	5674	135
4	34	5367	180	122	17	197	5413	137
5	57	4298	325	135	24	397	7654	105
6	48	3987	450	180	36	497	5672	142
7	65	4871	595	215	52	710	6437	187
8	75	5001	380	190	40	464	9254	175
9	37	5456	417	192	62	447	4536	153
10	73	2786	560	165	25	272	5543	160
11	88	8327	250	100	75	720	8674	175
12	74	2367	190	212	47	870	2413	157
13	27	8298	225	235	74	297	9654	125
14	88	9987	650	280	26	497	3672	132
15	16	6871	195	315	82	410	7437	127
16	95	2001	880	290	20	964	2254	195
17	26	9871	217	192	94	897	9654	173
18	95	8001	860	125	86	297	9672	190
19	56	2871	250	220	72	910	8437	185
20	75	4001	380	121	90	264	8254	172

We can analyze units based on data and analysis methods described in Section 3. *Table 3* shows the overall efficiency and stage efficiency based on the multiplication model of the input orientaion.

Table 3. Table results based on the multiplication model of the input orientation.

DMUS	s1 Olefin Unit	s2 Tetramer Unit	s3 Dodecylbenzene Unit	Overall Total Efficiency
DMU1	0.295500	0.668100	0.016600	0.202300
DMU2	0.255500	0.675200	0.009000	0.175400
DMU3	0.153400	0.813700	0.008500	0.126100
DMU4	0.130700	0.468300	0.018400	0.062600
DMU5	0.036200	0.563200	0.006200	0.021200
DMU6	0.255200	0.914800	0.012200	0.234200
DMU7	0.256000	0.636800	0.007600	0.168800
DMU8	0.311300	0.507500	0.006600	0.168400
DMU9	0.301400	1.000000	0.020200	0.299500
DMU10	0.322100	0.804400	0.011100	0.260200
DMU11	0.226300	0.272600	0.005900	0.066600
DMU12	0.087900	0.272200	0.018700	0.025800
DMU13	0.097300	0.707500	0.010300	0.071000
DMU14	0.146100	0.688500	0.009100	0.101400
DMU15	1.000000	1.000000	0. 17700	0.991900
DMU16	0.246700	1.000000	0.018100	0.245400
DMU17	0.909800	0.714100	0.014200	0.653200
DMU18	0.173200	0.881900	0.005400	0.152000
DMU19	1.000000	0.457800	1.574600	0.457000
DMU20	0. 00800	0..239200	0.009800	0.012500

Based on the findings, the studied units, 15 and 20, have the bests and worst conditions, respectively. *Fig. 7* shows the chain's overall efficiency and the stage efficiency in a graph.

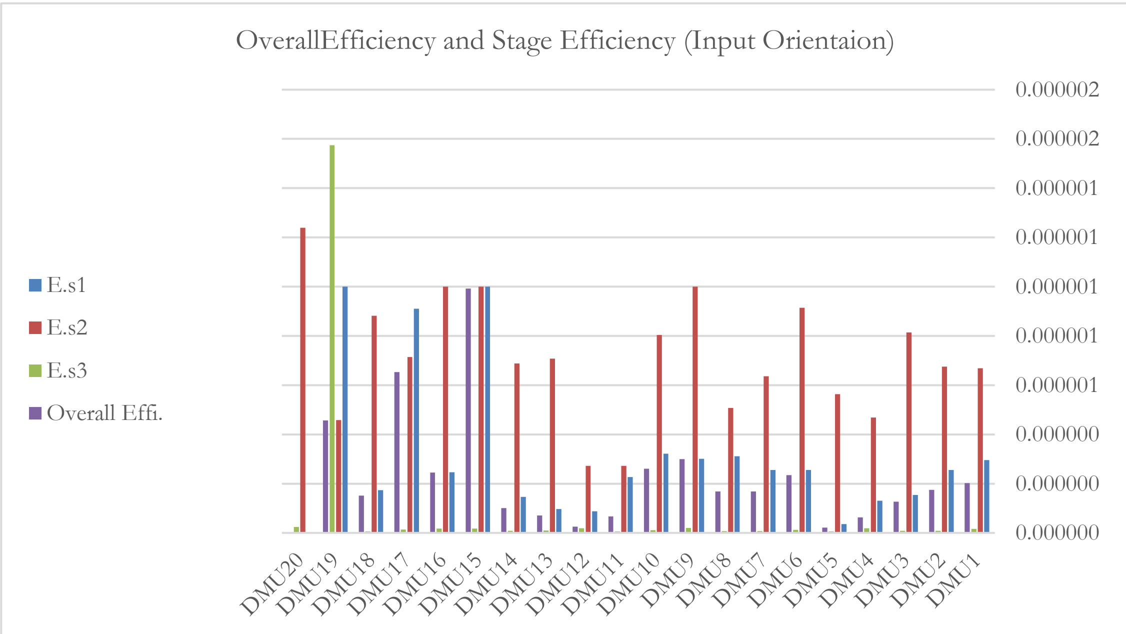

Fig. 7. Efficiency of the entire chain and stage efficiency.

By investigating units in the output orientaion according to the equations of Section 3, the overall efficiency and stage efficiency from the multiplication model of the output orientaion is:

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Table 4. Table results based on the multiplication model of the output orientation.

DMUS	s1 Olefin Unit	s2 Tetramer Unit	s3 Dodecylbenzene Unit	Overall Total Efficiency
DMU1	0.297053	0.52579	0.000242	0.105361
DMU2	0.257162	0.501555	0.000211	0.091121
DMU3	0.154257	0.554847	0.000208	0.067056
DMU4	0.130305	0.392881	0.000347	0.032143
DMU5	0.035933	0.398486	0.000241	0.011208
DMU6	0.256226	0.640369	0.000243	0.125873
DMU7	0.257838	0.433388	0.000267	0.089227
DMU8	0.313607	0.380474	0.000283	0.089066
DMU9	0.302115	0.730194	0.000246	0.155698
DMU10	0.324055	0.520806	0.000298	0.13488
DMU11	0.227211	0.190647	0.000538	0.034385
DMU12	0.08791	0.287836	0.000229	0.013317
DMU13	0.097619	0.722543	0.000164	0.038245
DMU14	0.146041	0.492417	0.000146	0.052429
DMU15	1	0.566669	0.00124	0.57501
DMU16	0.248806	0.664761	0.000207	0.127915
DMU17	0.915081	0.654236	0.000275	0.344542
DMU18	0.174679	0.504999	0.000464	0.080118
DMU19	1	0.430626	0.000257	0.228352
DMU20	0.16542	0.54652	0.000245	0.34210

According to the results in Stage 1 i.e., Olefin unit the DMU5 has the lowest score and DMU15 and DMU19 are efficient. In stage 2 DMU11 has the lowest scoreand DMU9 has the highest score but not efficient. The average score in stage 3 is lower than the other two stages and the highest score in this stage is for DMU11 and the lowest score is for DMU15. Based on the findings, the studied units, 15 and 12, have the bests and worst conditions, respectively (note that the values of the above table are the reverse of values obtained from the model. These values are between 0 and 1, and comparing them is easier, and they represent efficiency). See the below graph. *Fig. 8* shows the chain's overall efficiency and the stage efficiency.

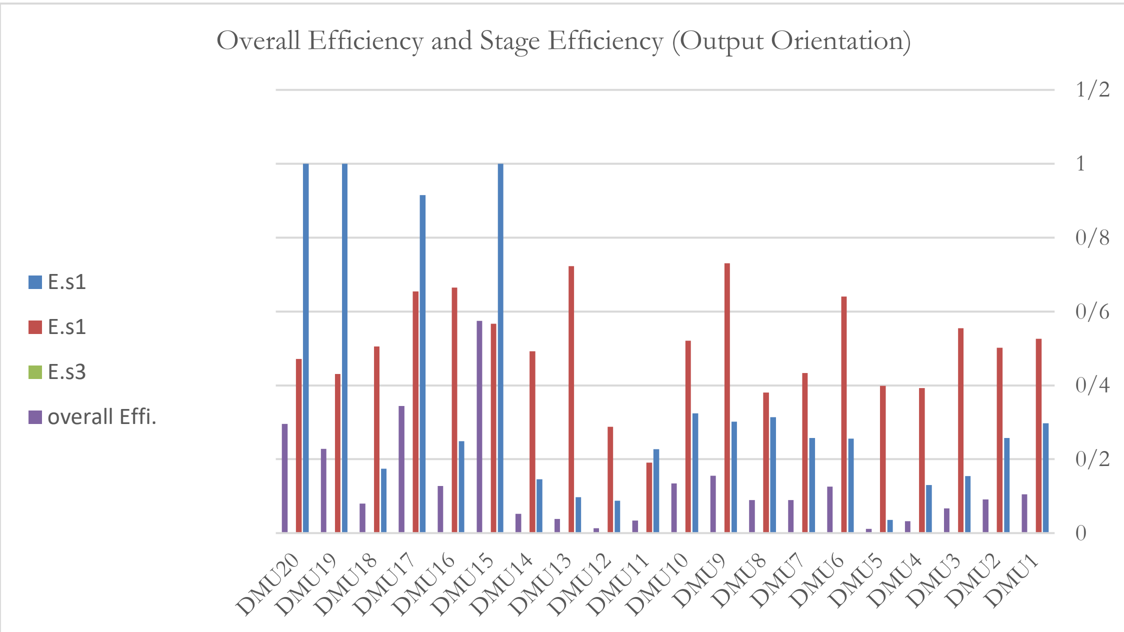

Fig. 8. Efficiency of the entire chain and stage efficiency.

If the evaluation is conducted based on the equations of aggregate efficiency, *Table 5* shows the aggregate efficiency and stage efficiency based on the multiplication model of the input orientaion.

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Table 5. Table results based on the aggregate efficiency of the multiplication model of the input orientation.

DMUs	s1 Olefin Unit	s2 Tetramer Unit	s3 Dodecylbenzene Unit	Agregate Total Efficiency
DMU1	0.6682	0.2707	0.0545	0.6618
DMU2	0.6561	0.2845	0.0334	0.6485
DMU3	0.2895	0.1452	0.0192	0.2854
DMU4	0.2899	0.0487	0.0840	0.2861
DMU5	0.0563	0.2203	0.0162	0.0553
DMU6	0.5311	0.3133	0.0273	0.5247
DMU7	0.5825	0.1402	0.0257	0.5742
DMU8	0.5470	0.2241	0.0210	0.5381
DMU9	1.0000	0.2263	0.0666	0.9884
DMU10	1.0000	0.4836	0.0424	0.9898
DMU11	0.4829	0.0403	0.0418	0.4743
DMU12	0.0995	0.1212	0.0712	0.0982
DMU13	0.1543	0.0975	0.0220	0.1509
DMU14	0.5508	0.0837	0.0487	0.5414
DMU15	1.0000	0.0615	0.0175	0.9822
DMU16	0.9156	0.4185	0.0670	0.9094
DMU17	1.0000	0.0726	0.0213	0.9770
DMU18	0.5324	0.1116	0.0186	0.5218
DMU19	1.0000	0.2100	0.0234	0.9846
DMU20	1.0000	0.2370	0.0254	0.9854

Based on the findings, the studied units, 15 and 5, have the bests and worst conditions, respectively. *Fig. 9* shows the chain's overall efficiency and the stage efficiency in a graph.

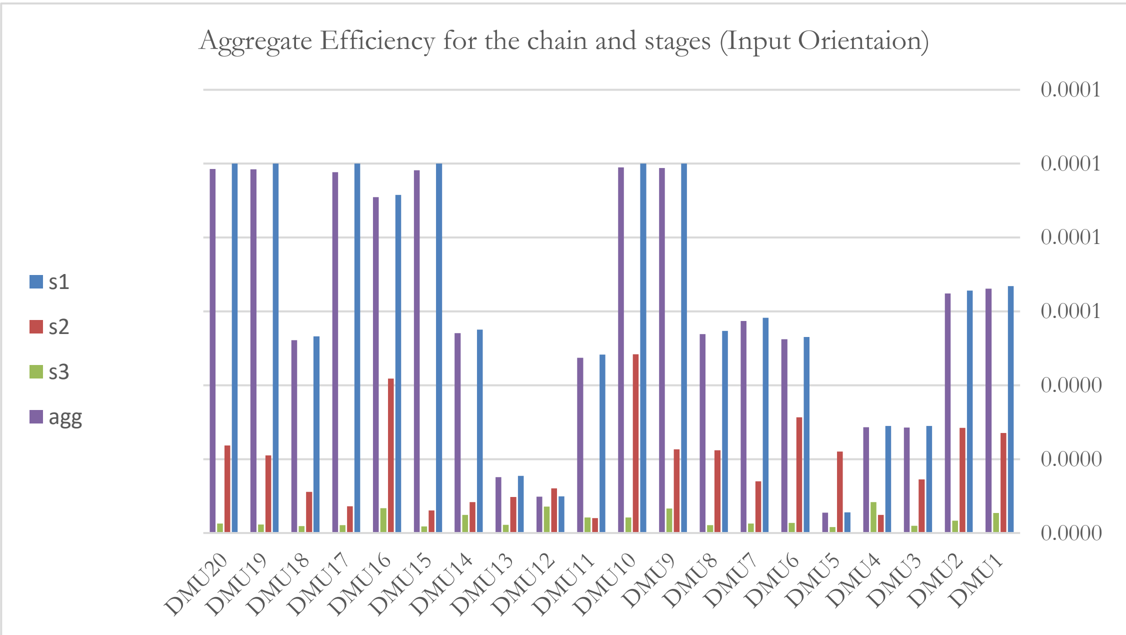

Fig. 9. Aggregate efficiency of the chain and stage efficiency.

If the evaluation is conducted based on the equations of aggregate efficiency, *Table 6* shows the aggregate efficiency and stage efficiency based on the multiplication model of the input orientaion.

Table 6. Table results based on the aggregate efficiency of the multiplication model of the input orientation.

DMUs	s1 Olefin Unit	s2 Tetramer Unit	s3 Dodecylbenzene Unit	Aggregate Total Efficiency
DMU1	0.297053	0.668047	0.082275	0.125521
DMU2	0.257162	0.675037	0.051482	0.10824
DMU3	0.154257	0.813736	0.067213	0.073573
DMU4	0.130305	0.46834	0.293703	0.043523
DMU5	0.035933	0.563095	0.280041	0.014176
DMU6	0.256226	0.914829	0.052005	0.131082
DMU7	0.257838	0.636699	0.044825	0.108016
DMU8	0.313607	0.507331	0.039077	0.116572
DMU9	0.302115	1	0.067347	0.155674
DMU10	0.324055	0.804505	0.042791	0.14906
DMU11	0.227211	0.272361	0.088062	0.053503
DMU12	0.08791	0.272087	0.724795	0.020727
DMU13	0.097619	0.706864	0.14569	0.04436
DMU14	0.146041	0.688326	0.090032	0.061836
DMU15	1	1	0.017858	0.574317
DMU16	0.248806	1	0.073677	0.127894
DMU17	0.915081	0.6943	0.021754	0.402804
DMU18	0.174679	0.881834	0.035708	0.084963
DMU19	1	0.458064	0.023763	0.312764
DMU20	1	0.438001	0.024341	0.321823

Based on the findings, the studied units, 15 and 12, have the bests and worst conditions, respectively. The below figure shows the chain's overall efficiency and the stage efficiency. Note that the values of the above table are the reverse of values obtained from the model. These values are between 0 and 1, and comparing them is easier, and they represent efficiency.

According to *Fig. 10*, a comparison between the aggregate efficiency and the stage efficiency can be made.

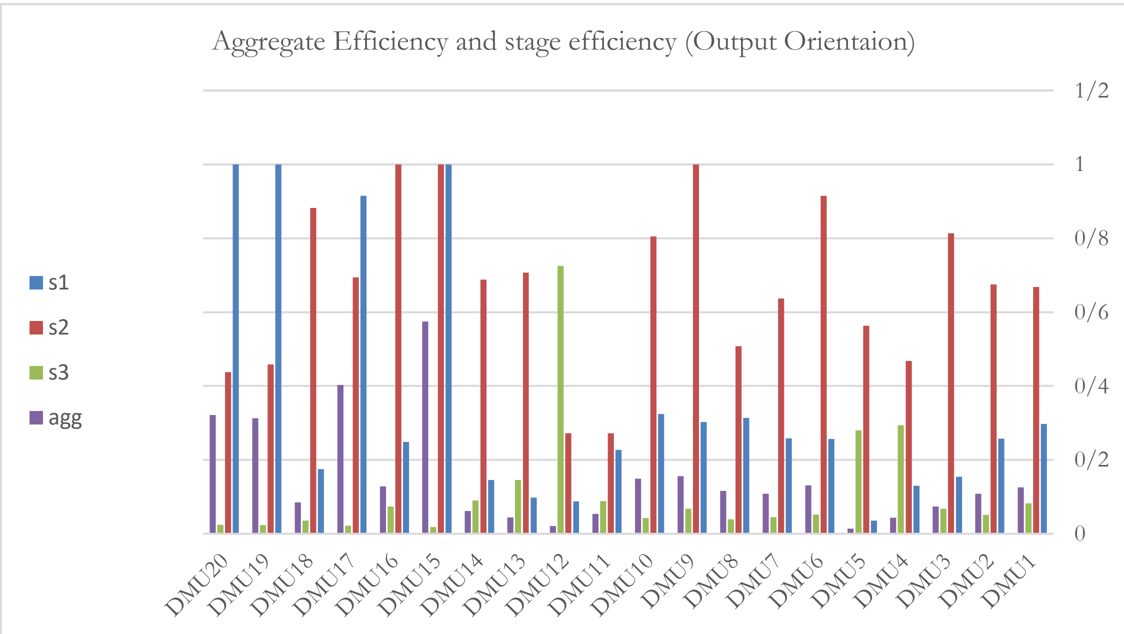

Fig. 10. Aggregate efficiency and stage efficiency.

5 | Conclusion

The studies in the literature of DEA for performance evaluation of petrochemicals sector, the existing competitive situation in petrochemicals sector and increasing negotiating power among customers in internal and international environment indicate importance of paying attention to this sector.

In this research, the efficiency and workability of petrochemical units were investigated. This study aimed to consider the network structure of the efficiency measurement by considering reversible equations. The important issue in the efficiency measurement and using data envelopment analysis is the correct selection of indicators and the contextual factors to create a meaningful model. To do this, in this paper the processes of the production system in olefin, tetramer, and dodecylbenzene were specified, and reviewing the literature and interviewing the expert the most suitable indicators were gathered for performance evaluation. During this research, there were many limitations in data collection, index definition, and conducting research. The limitations were the unavailability of information on petrochemical units to evaluate performance and ranking units. Also, inefficient and efficient units were not named due to data confidentiality and limitation in collecting them. Notably, the findings of this research are not validated permanently and are limited to the time of data collection.

The purpose of this study is to evaluate the performance of 20 companies involved in petrochemical production. The methodology used in this research can be used in all gas and oil refineries in addition to the gas transmission regions, etc. Also, the weakness and problems in production in triple processes of petrochemical were identified to find solutions, and units with better situations were supported and encouraged.

Future studies based on the outcomes of this research are suggested below:

The utilized methodology of this research can be implemented in all gas and oil refineries in addition to the gas transmission regions, etc.

We can also evaluate the supply chain of the petrochemical units. The supply chain evaluation should correspond to the strategy of these organizations. First, a strategy is written for petrochemical units. Then, they are designed according to the network structure and are finally examined. Modeling should be conducted by considering the hybrid orientation of the SBM model to evaluate the overall and stage efficiency of each component of the chain.

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