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# Ranking Efficient DMUs Using the Tchebycheff Norm with Fuzzy Data in DEA

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**A B S T R A C T**

In many real applications, the data of production processes cannot be precisely measured. Hence the input and output of Decision Making Units (DMUs) in Data Envelopment Analysis (DEA) may be imprecise or fuzzy-numbered. In original DEA models, inputs and outputs are measured by exact values on a ratio scale, therefore conventional DEA can't easily measure the performance of DMUs and rank them. The researchers have introduced mane deferent model for ranking DMUs by fuzzy number. In this paper, we proposed a new method by using the Tchebycheff norm for ranking DMUs with fuzzy data. We explain our method by numerical example with the triangular fuzzy number.

**Keywords**: Data envelopment analysis, Ranking, Tchebycheff norm, Fuzzy system.

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## 1. Introduction

Data Envelopment Analysis (DEA) is a powerful approach in measuring efficiency of decision making units with multiple inputs and outputs. DEA was introduced by Charnes et al. [4] and extended by Banker et al. [2]. DEA successfully divides them into two categories, efficient DMUs and inefficient DMUs. One of the main challenges associated with the application of DEA is the difficulty in quantifying some of these input and output factors. In other words, a key to success of the DEA approach is the accurate measure of all factors, including inputs and outputs. However, a production process usually involves complicated inputs and outputs; in that many factors are very difficult to measure in a precise manner. This makes an approach which is able

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to deal with inexact numbers or numbers in ranges and desirable. One way represented the uncertain values by membership functions of the fuzzy set theory [20]. When some observations are fuzzy, the goal and constraints in the decision process become fuzzy as well. We can find several fuzzy approaches to the assessment of efficiency in the DEA literature. Sengupta [16] was the first to introduce a fuzzy mathematical programming approach where the constraints and objective function are not satisfied crisply. In his paper, a generalization of DEA models is considered for DMUs with multiple inputs and a single output. A fuzzy k-means clustering approach as a means of identifying unusual and/or extreme efficiency behavior is proposed in [14]. In Cooper et al. [5], the imprecise DEA method has been developed. This method permits mixture of imprecisely data within specified bounds and exactly known data. Kao and Liu [11] developed a method to find the membership functions of the fuzzy efficiency scores when some observations are fuzzy numbers. After evaluation DMUs by DEA model may be more than one DMU be efficient. For elimination this problem the ranking models are proposed. Some methods for ranking efficient DMUs with crisp data are proposed. Anderson and Petersen [1] evaluated that a DMUs efficiency possibly exceeds the conventional score 1.0, by comparing the DMU evaluated with a linear combination of other DMUs, while excluding the observations of the DMU evaluated. They tried to discriminate between these efficient DMUs, by using different efficiency scores larger than 1.0. Mehrabian et al. [12] presented the popular of these methods. These methods would have some deficiencies if data have certain structures. There are some methods based on norms. Jahanshahloo et al. [10] introduced an $L_1$-norm approach that removes some deficiencies arising from AP and MAJ, but that cannot rank non-extreme DMUs. Also Balf et al. [13] used the Tchebycheff norm for ranking DMUs. All the models that were proposed for ranking DMUs with crisp data aren´t able to ranking DMUs whit fuzzy data, therefore researchers tried to find models for ranking fuzzy DMUs. Jahanshahloo et al. [9] proposed ranking DMUs by $\mathrm{L}_1$-norm with fuzzy data in DEA. In this paper, by considering fuzzy DMUs we propose a new method based on Tchebycheff norm for ranking DMUs. Our proposed model is written based on domination between under evaluation DMU and other DMUs those are belong to the $T_C'$ that $T_C'$ obtained by omitting under evaluated DMU from $\mathrm{T_C}$ (Production Possibility Set (PPS) of CCR model). This paper consists of the following sections: Preliminary of fuzzy is presented in Section 2, review of ranking models is introduced in Section 3. In Section 4 we present our model, finally, an example with fuzzy data and then the conclusion be given.

## 2. Preliminaries

### 2.1. Fuzzy Numbers

**Definition 1.** A fuzzy number is a fuzzy set like $\mu : R \to I\ [0,1]$ which satisfies [8]:

- $\mu$ is upper semi-continuous.
- $\mu$(x) = 0 outside some interval $[a, d]$.
- There are real numbers a, b such that $a \le b \le c \le d$ and

- $\mu$ (x) is monotonic increasing on $[a, b]$,
- $\mu$ (x) is monotonic decreasing on $[c, d]$,
- $\mu$ (x) = 1, $b \le x \le c$ .

## 2.2. Distance Measure for Fuzzy Numbers

A fuzzy set $A = (a_1, a_2, a_3, a_4)$ is called a generalized left right fuzzy numbers (GLRFN) if its membership function satisfy the following (see [18]):

$$\mu(x) = \begin{cases} L(\frac{a_2 - x}{a_2 - a_1}) & , a_1 \le x \le a_2 \\ 1 & , a_2 \le x \le a_3 \\ R(\frac{x - a_3}{a_4 - a_3}) & , a_3 \le x \le a_4 \\ 0 & , \quad else \end{cases} \tag{1}$$

Where L and R, are strictly decreasing functions defined on [0, 1] and satisfying the conditions:

$$L(x) = R(x) = 1 \quad if \quad x \le 0, \tag{2}$$

$$L(x) = R(x) = 0 \quad if \quad x \ge 0. \tag{3}$$

For $a_2 = a_3$, we have the classical definition of Left Right Fuzzy Numbers (LRFN) of [7]. As described by [3], Triangular Fuzzy Numbers (TFN) are special cases of GLRFN with $L(x) = R(x) = 1 - x$ and $a_2 = a_3$. A triangular fuzzy number is denoted as $\tilde{A}= (a_1,a_2,a_3)$, see Figure 1.

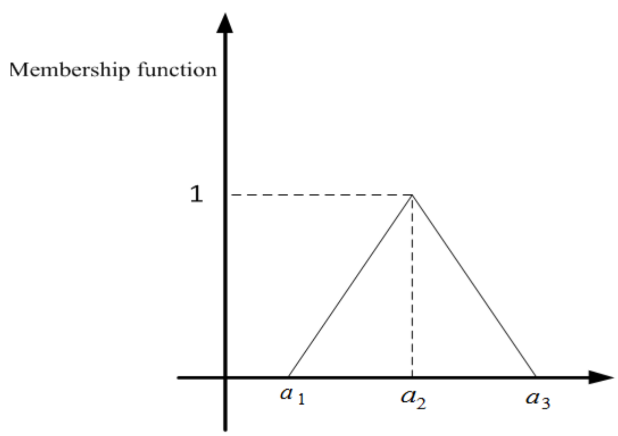


**Figure 1. Triangular fuzzy number.**

The membership function of a triangular fuzzy number is express as following [3]:

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1} & a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2} & a_2\le x\le a_3\\ 0, & \text{Otherwise.}\end{cases} \tag{4}$$

**Definition 2.** ($\alpha$-level set or $\alpha$-cut). The $\alpha$-cut of a fuzzy set $\tilde{A}$ is a crisp subset of X and is denoted by $\left[\tilde{A}\right]_{\alpha}=\left\{x\,|\,\mu_{\tilde{A}}(x)\ge\alpha\right\}$ [7].

A GLRFN A is denoted as $A=(a_1,a_2,a_3,a_4)_{L^A-R^A}$ and an $\alpha$-level interval of fuzzy number A as

$$\begin{aligned}A(\alpha)&=(A^l(\alpha),A^u(\alpha))_{LR}\\ A^l(\alpha)&=a_2-(a_2-a_1)L_A^{-1}(\alpha)\\ A^u(\alpha)&=a_3+(a_4-a_3)R_A^{-1}(\alpha).\end{aligned} \tag{5}$$

Let F be the family of the fuzzy numbers on R.

**Definition 3.** For $\tilde{A},\tilde{B}\in F$, define the signed distance of $\tilde{A},\tilde{B}$ as follows (see [18]):

$$d(A,B)=\int_0^1 w(\alpha)[(A^l(\alpha)-B^l(\alpha))+(A^u(\alpha)-B^u(\alpha))]d\alpha\,. \tag{6}$$

Here, $w(\alpha)$ is weighting function that $w:[0,1]\to[0,1]$. If $\int_0^1 w(\alpha)d\alpha=\frac{1}{2}$ then we say that $w(\alpha)$ is a regular function.

**Definition 4.** For $\tilde{A},\tilde{B}\in F$, define the ranking of $\tilde{A},\tilde{B}$ by saying (see [19]):

$$\begin{aligned}&d(\tilde{A},\tilde{B})\rangle 0 \quad if \quad \tilde{A}\succ\tilde{B}\\ &d(\tilde{A},\tilde{B})\langle 0 \quad if \quad \tilde{A}\prec\tilde{B}\\ &d(\tilde{A},\tilde{B})=0 \quad if \quad \tilde{A}\approx\tilde{B}.\end{aligned} \tag{7}$$

**Definition 5.** Let $\mu$ be a fuzzy number with $\alpha$-cut representation $(L_\mu(0),R_\mu(0))$ and let $w:[0,1]\to[0,1]$ be a reducing function. Then the value of $\mu$ (with respect to w) is (see [6]):

$$Val_w(\mu)=\int_0^1 w(\alpha)[L_\mu(\alpha)+R_\mu(\alpha)]d\alpha\,. \tag{8}$$

**Property 1.** For a trapezoidal fuzzy number T = ($a_1$, $a_2$, $a_3$, $a_4$) and $w(\alpha)=\alpha$ it is easy to show that:

$$Val(T)=\frac{a_3+a_2}{2}+\frac{[(a_4-a_3)-(a_2-a_1)]}{6}\,. \tag{9}$$

**Property 2.** T is a crisp interval, $a_4 = a_3$, $a_2 = a_1$. Obviously, $a_4 - a_3 = a_2 - a_1 = 0$ and, hence:

$$Val(T) = \frac{a_3 + a_2}{2}. \tag{10}$$

**Property 3.** T is a triangular fuzzy number, $a_2 = a_3 = v$, the vertex of the triangle. In this case

$$Val(T) = v + \frac{[(a_4 - v) - (v - a_1)]}{6}. \tag{11}$$

## 2.3. DEA Methodology

Consider *n*, DMUs use m input to produce s output. Let $x_{ip}$ $(\geq 0)$ represent input *i* of $DMU_p (p \in \{1, \ldots, n\})$ and $y_{ip}$ $(\geq 0)$ represents output *r* of it. Its actual point of operation is given by $(x_p, y_p) = (x_{1p}, \ldots, x_{mp}, y_{1p}, \ldots, y_{rp})$ and the projected point of $DMU_p$ is given by $(X, Y) = \left(\sum_{j=1}^{n} \lambda_j x_{1j}, \ldots, \sum_{j=1}^{n} \lambda_j x_{mj}, \sum_{j=1}^{n} \lambda_j y_{1j}, \ldots, \sum_{j=1}^{n} \lambda_j y_{rj}\right)$. The Production Possibility Sets (PPS) $T_c$ is defined as following.

$$T_c = \left\{ (X, Y) | X \geq \sum_{j=1}^{n} \lambda_j x_j, \; Y \leq \sum_{j=1}^{n} \lambda_j y_j, \lambda_j \geq 0, \; j = 1, .., n \right\}. \tag{12}$$

For evaluation of the efficiency of $DMU_p$, the envelopment form of CCR model in the input oriented case is as follows model [4].

$$\begin{aligned}
& Min\ \theta - \varepsilon \left( \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \ldots, s \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n \\
& s_i^- \geq 0, \quad i = 1, \ldots, m \\
& s_r^+ \geq 0, \quad r = 1, \ldots, s.
\end{aligned} \tag{13}$$

**Definition 6.** $DMU_p$ is efficient in model (13) if and only if $\theta^* = 1, s^{-*} = 0$ *and* $s^{+*} = 0$ (* means the optimal solution).

And its dual, the multiplier form of CCR model in the input oriented case is as follows [4]:

$$
\begin{aligned}
& Max \quad \sum_{r=1}^{s} u_r y_{rp} \\
& s.t. \\
& \sum_{i=1}^{m} v_i x_{ip} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j = 1,\dots,n \\
& u_r \ge \varepsilon \qquad r = 1,\dots,s \\
& v_i \ge \varepsilon \qquad s = 1,\dots,m \ .
\end{aligned}
\tag{14}
$$

## 3. Review of Ranking Models

In this subsection, we are going to review AP and MAJ and Tchebycheff norm ($L_\infty$) ranking models in data envelopment analysis.

### 3.1. AP Model

Anderson and Peterson [1] proposed the supper efficiency model. They omitted the efficient DMU from the PPS, $T_C$ and Ran CCR model for other units to rank them. Their proposed model is:

$$
\begin{aligned}
& Min \quad \theta \\
& s.t. \\
& \sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip} \quad , \quad i = 1,\dots,m \\
& \sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} \quad , \quad r = 1,\dots,s \\
& \lambda_j \ge 0 \quad , \quad j = 1,\dots,n \\
& s_i^- \ge 0 \quad , \quad i = 1,\dots,m \\
& s_r^+ \ge 0 \quad , \quad r = 1,\dots,s.
\end{aligned}
\tag{15}
$$

For efficient units, $\theta^* \ge 1$ and for inefficient units $0 < \theta^* < 1$ . This model has two drawbacks. (see [15] ):

- AP model may be inefficient for special data in input oriented case.
- This model is unstable for some DMUs, which one of their data components is near to zero.

### 3.2. MAJ Model

To solve the important drawbacks of AP models, Mehrabian et al. [12], proposed another model for ranking efficient units. Their proposed model is:

$$\begin{aligned} & Min \quad 1+w \\ & s.t. \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j x_{ij} \geq x_{ip} + w \quad , \; i=1,\ldots,m \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j y_{rj} \geq y_{rp} \quad , \; r=1,\ldots,s \\ & \lambda_j \geq 0 \;\; , \;\; j=1,\ldots,n. \end{aligned} \tag{16}$$

The necessary and sufficient conditions for feasibility of MAJ model is that in evaluating of $\text{DMU}_p$, or $y_{rp} = 0, r = 1, \ldots, s$ or there exists $\text{DMU}_j$, $j \neq p$ such that $y_{rj} \neq o$ (see [12]).

### 3.3. Tchebycheff Norm Model

Tavares and Antunes [17] were suggested a model for evaluate the efficiency, by Using L1-norm. They have been used Tchebycheff norm in the objective function of ADD model. For ranking extreme efficient DMUs Balf et al. [13] used the Tchebycheff norm for ranking efficient DMUs as follows:

$$\begin{aligned} & Min \;\; V_p \\ & s.t. \\ & V_p \geq \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j x_{ij} - x_{ip} \quad , \; i=1,\ldots,m \\ & V_p \geq y_{rp} - \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j y_{rj} \quad , \; r=1,\ldots,s \\ & \lambda_j \geq 0 \;\; , \;\; j=1,\ldots,n. \end{aligned} \tag{17}$$

Note that the model (15) is independent of oriented (input-oriented and output-oriented), therefore, it is a superiority over other existence methods.

## 4. Our Proposed Model

In this section, we suppose that input and output of DMUs are fuzzy.

**Definition 7.** Assume that there are a set of n DMUs. Each $DMU_j \, (j=1,\ldots,n)$ has $m$ inputs and $s$ different output, which are denoted as $\tilde{x}_{ij} \, (i=1,2,\ldots,m)$ and $\tilde{y}_{rj} \, (r=1,2\ldots,s)$, respectively. We assume all input and output are fuzzy data. For ranking extreme efficient $DMU_P$, first it will be removed from PPS ($T_C$) and then new PPS ($T'_C$) is defined as follows:

$$\tilde{T}'_C = \left\{ \left(\tilde{X}, \tilde{Y}\right) | \tilde{X} \geq \sum_{j=1, j\neq p}^{n} \lambda_j \tilde{X}_j \quad , \tilde{Y} \leq \sum_{j=1, j\neq p}^{n} \lambda_j \tilde{Y}_j \ , \ \lambda_j \geq 0 \ , \ \ j = 1, \dots, n \right\}. \tag{18}$$

It is specifying that if $DMU_P$with coordinates $\left(\tilde{X}_P, \tilde{Y}_P\right)$ be inside Production Possibility Set (PPS) then we have:

$$\exists \lambda \geq 0 \qquad s.t. \ \sum_{j=1}^{n} \lambda_j \tilde{X}_j \leq \tilde{X}_p \ \text{ and } \sum_{j=1}^{n} \lambda_j \tilde{Y}_j \geq \tilde{Y}_p \ . \tag{19}$$

But if the $DMU_P$ (under evaluation unit) lie outside PPS, then we have:

$$\left( \sum_{j=1}^{n} \lambda_j \tilde{X}_j \geq \tilde{X}_p \ \ , \sum_{j=1}^{n} \lambda_j \tilde{Y}_j \leq \tilde{Y}_p \ , \lambda_j \geq 0, j = 1, \dots, n \right). \tag{20}$$

In other word

$$\exists i \ \ , \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} > \tilde{x}_{ip} \ \ or \ \ \exists r \ \ , \sum_{j=1}^{n} \lambda_j \tilde{y}_j < \tilde{y}_{rp} \, . \tag{21}$$

Now, suppose $DMU_P$ is outside of PPS. Hence, we want to obtain a point of PPS which dominated by $DMU_P$. Thus, we have (22):

$$\exists \lambda \geq 0 \ s.t. \ \sum_{j=1}^{n} \lambda_j \tilde{X}_j \geq \tilde{X}_p \ \ , \sum_{j=1}^{n} \lambda_j \tilde{Y}_j \leq \tilde{Y}_p \, . \tag{22}$$

So, the following model is suggested for ranking $DMU_P$.

$$\begin{aligned} &\left\| \tilde{P} - \hat{P} \right\|_{\infty} \\ &s.t. \ \ \hat{P} \in \tilde{T}'_C \ . \end{aligned} \tag{23}$$

Where $\tilde{P} = (\tilde{x}_p, \tilde{y}_p)$ is the point under evaluation and $\hat{P} = \left( \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j \, \tilde{x}_{ij}, \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j \, \tilde{y}_{rj} \right)$ is a point in $\tilde{T}'_C$. Note that model (23) is written based on domination between $\tilde{P}$ and$\hat{P}$. The objective function of the model (23) is nonlinear. Following, after manipulation, model (23) is converted in a linear model, which can be easily solved. By definition of infinity norm ($L_\infty$) that $\left\|\tilde{P} - \hat{P}\right\|_\infty = Min \ Max\left\{\left|\tilde{P} - \hat{P}\right|\right\}$ the objective of (23) can be rewritten as follows:

$$\begin{aligned} &Min \, Max \left\{ \left| \tilde{P} - \hat{P} \right| \right\} \\ &s.t. \ \ \hat{P} \in \tilde{T}'_C \ . \end{aligned} \tag{24}$$

With using of Definition (4) and values of $\tilde{P}$ and $\hat{P}$ the objective of model (24) are changed as follows:

$$Min\,Max\left\{\left|\tilde{P}-\hat{P}\right|\right\}=Min\,Max\left\{d\left(\tilde{P},\hat{P}\right)\right\}.$$

$$=Min\,Max\left\{d\left(\tilde{x}_{ip},\sum_{j=1,j\neq p}^{n}\lambda_j\tilde{x}_{ij}\right)_{i=1,\ldots,m},d\left(\tilde{y}_{rp},\sum_{j=1,j\neq p}^{n}\lambda_j\tilde{y}_{rj}\right)_{r=1,\ldots,s}\right\} \tag{25}$$

Therefore we have:

$$\begin{aligned}
&Min\,Max\left\{d\left(\tilde{x}_{ip},\sum_{j=1,j\neq p}^{n}\lambda_j\tilde{x}_{ij}\right)_{i=1,\ldots,m},\ d\left(\tilde{y}_{rp},\sum_{j=1,j\neq p}^{n}\lambda_j\tilde{y}_{rj}\right)_{r=1,\ldots,s}\right\}\\
&d\left(\sum_{j=1,j\neq p}^{n}\lambda_j\tilde{x}_{ij},\tilde{x}_{ip}\right)\geq 0 \qquad i=1,\ldots,m\\
&d\left(\tilde{y}_{rp},\sum_{j=1,j\neq p}^{n}\lambda_j\tilde{y}_{rj}\right)\geq 0 \qquad r=1,\ldots,s\\
&d\left(\tilde{x}_{ip},0\right)\geq 0 \qquad i=1,\ldots,m\\
&d\left(\tilde{y}_{rp},0\right)\geq 0 \qquad r=1,\ldots,s\\
&\lambda_j\geq 0, \qquad j=1,\ldots,n,\ j\neq p.
\end{aligned} \tag{26}$$

Also, if we use of Definition (3) then we will have crisp $L_\infty$-norm model as (27).

$$
\begin{aligned}
& Min\ \ Max \left\{ \int_0^1 w(\alpha)\left[\left(x_{ip}^{l}(\alpha)-\sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}^{l}\right)+\left(x_{ip}^{u}(\alpha)-\sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}^{u}\right)\right]_{i=1,\dots,m} d\alpha \right. \\
& \left. ,\int_0^1 w(\alpha)\left[\left(y_{rp}^{l}(\alpha)-\sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}^{l}\right)+\left(y_{rp}^{u}(\alpha)-\sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}^{u}\right)\right]_{r=1,\dots,s} d\alpha \right\} \\
& s.t. \\
& \int_0^1 w(\alpha)\left[\left(\sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}^{l}-x_{ip}^{l}(\alpha)\right)+\left(\sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}^{u}-x_{ip}^{u}(\alpha)\right)\right] d\alpha \geq 0 \qquad i=1,\dots,m \\
& \int_0^1 w(\alpha)\left[\left(y_{rp}^{l}(\alpha)-\sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}^{l}\right)+\left(y_{rp}^{u}(\alpha)-\sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}^{u}\right)\right] d\alpha \geq 0 \qquad r=1,\dots,s \\
& \int_0^1 w(\alpha)\left[\left(x_{ip}^{l}(\alpha)-0\right)+\left(x_{ip}^{u}(\alpha)-0\right)\right] d\alpha \geq 0 \qquad i=1,\dots,m \\
& \int_0^1 w(\alpha)\left[\left(y_{rp}^{l}(\alpha)-0\right)+\left(y_{rp}^{u}(\alpha)-0\right)\right] d\alpha \geq 0 \qquad r=1,\dots,s \\
& \lambda_j \geq 0 \qquad j=1,\dots,n\ \ ,\ j\neq p.
\end{aligned}
\tag{27}
$$

We can simply reformulated model (27) with using Definition (5) as model (28).

$$
\begin{aligned}
& Min\ \ Max \left[\left(Val\left(\tilde{x}_{ip}\right)-\sum_{j=1,j\neq p}^{n}\lambda_j Val\left(\tilde{x}_{ij}\right)\right),\left(Val\left(\tilde{y}_{rp}\right)-\sum_{j=1,j\neq p}^{n}\lambda_j Val\left(\tilde{y}_{rj}\right)\right)\right] \\
& s.t. \\
& \left(\sum_{j=1,j\neq p}^{n}\lambda_j Val\left(\tilde{x}_{ij}\right)-Val\left(\tilde{x}_{ip}\right)\right)\geq 0 \qquad i=1,..,m \\
& \left(Val\left(\tilde{y}_{rp}\right)-\sum_{j=1,j\neq p}^{n}\lambda_j Val\left(\tilde{y}_{rj}\right)\right)\geq 0 \qquad r=1,..,s \\
& Val\left(\tilde{x}_{ij}\right)\geq 0 \qquad i=1,\dots,m \\
& Val\left(\tilde{y}_{rj}\right)\geq 0 \qquad r=1,\dots,s \\
& \lambda_j \geq 0\ \ , \qquad j=1,\dots,n\ \ ,\ j\neq p.
\end{aligned}
\tag{28}
$$

If we consider

$$\gamma = Max\left\{\left(Val\left(\tilde{x}_{ip}\right) - \sum_{j=1, j\neq p}^{n} \lambda_j \ Val(\tilde{x}_{ij})\right), \left(Val\left(\tilde{y}_{rp}\right) - \sum_{j=1, j\neq p}^{n} \lambda_j Val\left(\tilde{y}_{rj}\right)\right)\right\} \tag{29}$$

then we have

$$\begin{aligned} &\gamma - \left(Val\left(\tilde{x}_{ip}\right) - \sum_{j=1, j\neq p}^{n} \lambda_j Val\left(\tilde{x}_{ij}\right)\right) \geq 0 \\ &\gamma - \left(Val\left(\tilde{y}_{rp}\right) - \sum_{j=1, j\neq p}^{n} \lambda_j Val\left(\tilde{y}_{rj}\right)\right) \geq 0. \end{aligned} \tag{30}$$

As was mentioned, we propose the final model (31).

$$\begin{aligned} &Min \quad \gamma \\ &\gamma - \left(Val\left(\tilde{x}_{ip}\right) - \sum_{j=1, j\neq p}^{n} \lambda_j Val\left(\tilde{x}_{ij}\right)\right) \geq 0 \\ &\gamma - \left(Val\left(\tilde{y}_{rp}\right) - \sum_{j=1, j\neq p}^{n} \lambda_j Val\left(\tilde{y}_{rj}\right)\right) \geq 0 \\ &\sum_{j=1, j\neq p}^{n} \lambda_j Val\left(\tilde{x}_{ij}\right) - Val\left(\tilde{x}_{ip}\right) \geq 0 \\ &Val\left(\tilde{y}_{rp}\right) - \sum_{j=1, j\neq p}^{n} \lambda_j Val\left(\tilde{y}_{rj}\right) \geq 0 \\ &Val\left(\tilde{x}_{ij}\right) \geq 0 && i = 1,\dots,m \\ &Val\left(\tilde{y}_{rj}\right) \geq 0 && r = 1,\dots,s \\ &\lambda_j \ \geq 0 \ , && j = 1,\dots,n \quad , \ j \neq p. \end{aligned} \tag{31}$$

In a similar method, we can write the model for Trapezoidal fuzzy number**.**

## 5. Numerical Example

In this section, we rank DMUs, when inputs and outputs of DMUs are triangular fuzzy number. This study selects a small example using data given by Jahanshahloo et al. [9]. We examine the proposed model for 30 branches of social Welfare bureau in Tehran which data of each of these branches was collected in different 18 periods. The conclusions obtained of these observations are explained in following tables. Table 1 introduces the input and output of DMUs.

**Table 1. Inputs and outputs.**

| Inputs | Outputs |
|---|---|
| (1) Number of the personals | (1) Number of the whole insured |
| (2) Number of the computers | (2) Number of the insurance policy |
| (3) The area of branch | (3) Number of the whole pensioner |
| (4) The amount of whole administrative | (4) The amount of income |

In Tables 2–5 the triangular fuzzy data of these branches is shown.

**Table 2. Fuzzy data of the first input and output.**

| DMU | min(I1) | average(I1) | max(I1) | min(O1) | average(O1) | max(01) |
|---|---|---|---|---|---|---|
| 1 | 90 | 94.9 | 100 | 55830 | 64411 | 86532 |
| 2 | 75 | 77.6 | 81 | 36216 | 36760.5 | 371798 |
| 3 | 75 | 77.1 | 80 | 24566 | 32588.7 | 39449 |
| 4 | 90 | 92.8 | 96 | 35469 | 36307.2 | 37722 |
| 5 | 88 | 90.9 | 93 | 34734 | 45726.9 | 56082 |
| 6 | 95 | 101 | 105 | 58344 | 68019.3 | 78574 |
| 7 | 89 | 93.7 | 100 | 32585 | 38983.9 | 42573 |
| 8 | 83 | 87.7 | 93 | 429 | 50852.4 | 63341 |
| 9 | 99 | 104.1 | 112 | 84531 | 90371.2 | 100220 |
| 10 | 102 | 105 | 111 | 46924 | 52505.7 | 61767 |
| 11 | 93 | 95.9 | 101 | 31554 | 39487.3 | 54521 |
| 12 | 76 | 78.5 | 79 | 26687 | 32757.7 | 39620 |
| 13 | 102 | 102.9 | 107 | 56144 | 60769.9 | 65026 |
| 14 | 82 | 87.5 | 91 | 80425 | 91556.4 | 96821 |
| 15 | 77 | 81.4 | 84 | 44305 | 50330.7 | 52856 |
| 16 | 87 | 88.8 | 91 | 39797 | 46576.2 | 49855 |
| 17 | 84 | 90.3 | 93 | 52923 | 70898.9 | 82923 |

| DMU | min(I1) | average(I1) | max(I1) | min(O1) | average(O1) | max(01) |
|---|---|---|---|---|---|---|
| 18 | 94 | 109.2 | 117 | 72553 | 85571.6 | 98678 |
| 19 | 89 | 96.5 | 103 | 46154 | 69124.8 | 89844 |
| 20 | 82 | 86.8 | 90 | 26927 | 31451.8 | 37990 |
| 21 | 68 | 70.6 | 73 | 27128 | 2805.2 | 29058 |
| 22 | 108 | 114.8 | 119 | 102175 | 111927.8 | 129100 |
| 23 | 75 | 79.7 | 86 | 27704 | 33378.3 | 36205 |
| 24 | 85 | 88.6 | 93 | 51345 | 57346 | 61760 |
| 25 | 96 | 98.7 | 103 | 72915 | 81096.7 | 88038 |
| 26 | 73 | 77.6 | 83 | 42887 | 44066.2 | 45396 |
| 27 | 100 | 106.9 | 112 | 78068 | 80831.3 | 83148 |
| 28 | 96 | 99.8 | 105 | 5865 | 69577.1 | 74218 |
| 29 | 67 | 71.6 | 76 | 38054 | 39330.97 | 40860 |
| 30 | 86 | 89.1 | 93 | 59846 | 62360 | 64784 |

**Table 3. Fuzzy data of the second input and output.**

| DMU | min(I2) | average(I2) | max(I2) | min(O2) | average(O2) | max(O2) |
|---|---|---|---|---|---|---|
| 1 | 83 | 85.1 | 87 | 17 | 39.1 | 71 |
| 2 | 88 | 92.3 | 95 | 0 | 18.5 | 35 |
| 3 | 85 | 87.5 | 89 | 11 | 21.9 | 47 |
| 4 | 93 | 93.8 | 96 | 10 | 31.2 | 59 |
| 5 | 83 | 85.2 | 87 | 9 | 33.2 | 50 |
| 6 | 97 | 97.3 | 98 | 0 | 14.7 | 33 |
| 7 | 90 | 92.5 | 97 | 47 | 97.5 | 130 |
| 8 | 92 | 93.1 | 95 | 11 | 21.8 | 34 |
| 9 | 84 | 92.5 | 101 | 0 | 57.2 | 111 |
| 10 | 95 | 96.6 | 100 | 9 | 32.2 | 60 |
| 11 | 78 | 79.4 | 82 | 81 | 201.8 | 268 |
| 12 | 89 | 90.7 | 93 | 11 | 25.6 | 36 |
| 13 | 103 | 105.9 | 111 | 27 | 44.9 | 77 |
| 14 | 92 | 96.7 | 102 | 23 | 40.1 | 66 |
| 15 | 92 | 93.2 | 95 | 6 | 17.3 | 29 |

| DMU | min(I2) | average(I2) | max(I2) | min(O2) | average(O2) | max(O2) |
|---|---|---|---|---|---|---|
| 16 | 85 | 85.4 | 86 | 13 | 29.4 | 49 |
| 17 | 104 | 104.4 | 106 | 14 | 20.9 | 38 |
| 18 | 91 | 93.7 | 96 | 13 | 26.2 | 49 |
| 19 | 95 | 98.4 | 102 | 11 | 20.5 | 34 |
| 20 | 100 | 100.4 | 101 | 20 | 60.1 | 325 |
| 21 | 88 | 90.9 | 95 | 0 | 18.6 | 36 |
| 22 | 120 | 124.1 | 128 | 31 | 49.7 | 77 |
| 23 | 100 | 100.2 | 101 | 12 | 29.8 | 49 |
| 24 | 91 | 93.3 | 95 | 35 | 62.2 | 74 |
| 25 | 90 | 90 | 90 | 40 | 58.7 | 96 |
| 26 | 81 | 84.7 | 88 | 11 | 25.3 | 40 |
| 27 | 101 | 103.2 | 106 | 26 | 47.3 | 89 |
| 28 | 87 | 95.7 | 99 | 0 | 69.3 | 130 |
| 29 | 77 | 79.8 | 86 | 13 | 28.7 | 49 |
| 30 | 90 | 92.1 | 94 | 10 | 28.2 | 47 |

**Table 4. Fuzzy data of the third input and output.**

| DMU | min(I3) | average(I3) | max(I3) | min(O3) | average(O3) | max(O3) |
|---|---|---|---|---|---|---|
| 1 | 4000 | 4000 | 4000 | 1117 | 1224.8 | 1350 |
| 2 | 2565 | 2565 | 2565 | 8385 | 8604.6 | 8919 |
| 3 | 1343 | 1343 | 1343 | 6588 | 6633.3 | 6775 |
| 4 | 1500 | 1500 | 1500 | 8080 | 9867.9 | 10821 |
| 5 | 1680 | 1680 | 1680 | 9493 | 9918.8 | 10345 |
| 6 | 3750 | 3750 | 3750 | 7434 | 8017.7 | 8752 |
| 7 | 3313 | 3313 | 3313 | 13010 | 14224.6 | 15569 |
| 8 | 1500 | 1500 | 1500 | 1490 | 1574.7 | 1661 |
| 9 | 1600 | 1600 | 1600 | 10206 | 11284.4 | 15402 |
| 10 | 1725 | 1725 | 1725 | 6608 | 7812.6 | 9868 |
| 11 | 1920 | 1920 | 1920 | 11996 | 12619.4 | 13317 |
| 12 | 4433 | 4433 | 4433 | 7422 | 7798.3 | 8303 |
| 13 | 2500 | 2500 | 2500 | 7178 | 7425.2 | 7936 |

| DMU | min(I3) | average(I3) | max(I3) | min(O3) | average(O3) | max(O3) |
|---|---|---|---|---|---|---|
| 14 | 2800 | 2800 | 2800 | 630 | 734.4 | 930 |
| 15 | 1630 | 1630 | 1630 | 10247 | 10321.7 | 10503 |
| 16 | 1127 | 1127 | 1127 | 7302 | 7824.9 | 8322 |
| 17 | 3400 | 3400 | 3400 | 4740 | 4972.1 | 5205 |
| 18 | 1304 | 1304 | 1304 | 4279 | 4918.9 | 5289 |
| 19 | 4206 | 4206 | 4206 | 825 | 1212.9 | 1636 |
| 20 | 1340 | 1340 | 1340 | 14144 | 14859.7 | 15907 |
| 21 | 1393 | 1393 | 1393 | 921 | 944.5 | 973 |
| 22 | 2191 | 2191 | 2191 | 252 | 2488.9 | 3577 |
| 23 | 2140 | 2140 | 2140 | 234 | 2123.7 | 2477 |
| 24 | 1231 | 1231 | 1231 | 10157 | 10399.9 | 10841 |
| 25 | 1960 | 1960 | 1960 | 4193 | 4837.4 | 5869 |
| 26 | 3375 | 3375 | 3375 | 560 | 609.8 | 658 |
| 27 | 2540 | 2540 | 2540 | 8769 | 9359.6 | 9857 |
| 28 | 1603 | 1603 | 1603 | 8762 | 12905.5 | 14155 |
| 29 | 2300 | 2300 | 2300 | 1405 | 1491.3 | 1567 |
| 30 | 2930 | 2930 | 2930 | 11143 | 12184.7 | 13100 |

**Table 5. Fuzzy data of the fourth input and output.**

| DMU | min(I4) | average(I4) | max(I4) | min(O4) | average(O4) | max(O4) |
|---|---|---|---|---|---|---|
| 1 | 14730450 | 72806300 | 147806940 | 145 | 340.7 | 933 |
| 2 | 38109920 | 71065182 | 139078952 | 65 | 270.2 | 486 |
| 3 | 28792550 | 57285753 | 133424069 | 113 | 362.5 | 604 |
| 4 | 24277018 | 237014821 | 2592824900 | 54 | 233.1 | 398 |
| 5 | 43355800 | 77979393 | 140338064 | 101 | 280.2 | 461 |
| 6 | 8425500 | 125978083 | 810861434 | 82 | 379.3 | 644 |
| 7 | 63947100 | 122.32981 | 225550119 | 154 | 284.6 | 495 |
| 8 | 14969393 | 81051611 | 135858469 | 54 | 283.2 | 634 |
| 9 | 61024310 | 135992312 | 329155695 | 179 | 329.2 | 616 |
| 10 | 10112577 | 73804835 | 178541011 | 117 | 228.8 | 394 |
| 11 | 29357984 | 92014601 | 153783450 | 37 | 286.7 | 511 |

| DMU | min(I4) | average(I4) | max(I4) | min(O4) | average(O4) | max(O4) |
|---|---|---|---|---|---|---|
| 12 | 23106000 | 75843333 | 159552000 | 124 | 178.2 | 245 |
| 13 | 17756770 | 209943256 | 766788858 | 185 | 421.7 | 1175 |
| 14 | 30082208 | 76319046 | 168961589 | 51 | 195 | 391 |
| 15 | 45601750 | 72848488 | 158581843 | 28 | 145.3 | 295 |
| 16 | 26931600 | 112174234 | 827138457 | 85 | 248.9 | 580 |
| 17 | 17927100 | 116156884 | 475649794 | 43 | 175.4 | 286 |
| 18 | 25153373 | 126784004 | 389185592 | 72 | 187.6 | 324 |
| 19 | 5504769 | 81625443 | 264454915 | 74 | 232.1 | 530 |
| 20 | 48310364 | 94422470 | 153138326 | 190 | 310.9 | 444 |
| 21 | 19453875 | 54336511 | 105553653 | 55 | 233.4 | 457 |
| 22 | 45229750 | 97020370 | 190535604 | 120 | 289.1 | 468 |
| 23 | 45992927 | 109259866 | 285338884 | 156 | 383 | 589 |
| 24 | 5099980 | 80768509 | 148103734 | 85 | 181.3 | 278 |
| 25 | 52110247 | 147643177 | 757564117 | 112 | 242 | 427 |
| 26 | 12922183 | 69260832 | 113626397 | 218 | 320.8 | 450 |
| 27 | 21300223 | 81704431 | 187012354 | 134 | 278.4 | 679 |
| 28 | 45363108 | 307643735 | 931530808 | 91 | 197.8 | 304 |
| 29 | 10617400 | 72823790 | 110697939 | 23 | 115.5 | 304 |
| 30 | 57568217 | 84698270 | 122615610 | 122 | 263.3 | 406 |

The result of solution of our proposed method and $L_1$norm proposed by Jahanshahloo et al. [9] are summarized in Table 6.

**Table 6. The result of ranking.**

| DMU | Efficiency of Tchebycheff norm | Efficiency of $L_1$ norm | DMU | Efficiency of Tchebycheff norm | Efficiency of $L_1$ norm |
|---|---|---|---|---|---|
| 1 | 23.64645 | -1×E+8 | 16 | 1.93E+02 | -3×E+8 |
| 2 | 2.53E+04 | -1×E+8 | 17 | 0.05945 | -2×E+8 |
| 3 | 1.27E+02 | -8×E+7 | 18 | 0.043462 | -2×E+8 |
| 4 | 2.35E+03 | -7×E+8 | 19 | 0.27487 | -1×E+8 |
| 5 | 9.641513 | -1×E+8 | 20 | 4.35E+03 | -1×E+8 |
| 6 | 0.026414 | -3×E+8 | 21 | 0.294917 | -8×E+7 |
| 7 | 4.62E+02 | -2×E+8 | 22 | 8.78E+02 | -1×E+8 |
| 8 | 9.52E+02 | -1×E+8 | 23 | 19.37849 | -2×E+8 |
| 9 | 2.12E+02 | -2×E+8 | 24 | 2.91E+02 | -1×E+8 |
| 10 | 2.12E+02 | -1×E+8 | 25 | 3.359514 | -3×E+8 |
| 11 | 1.14E+02 | -1×E+8 | 26 | 0.066202 | -9×E+7 |
| 12 | 0.475155 | -1×E+8 | 27 | 0 | -1×E+8 |
| 13 | 14.26789 | -3×E+8 | 28 | 1.98E+02 | -5×E+8 |
| 14 | 0.044436 | -1×E+8 | 29 | 0.125881 | -9×E+7 |
| 15 | 0.054442 | -1×E+8 | 30 | 22.24281 | -1×E+8 |

You can see that in this table, the 2th and 20st DMUs have the best grade (ranking score) and the 27th DMU has the worst one by our proposed method. And by using $L_1$ norm you see that in this table the 3rd and 21st DMUs have the best grade (ranking score) and the 4th DMU has the worst one. Although the result of tow model is different, this does not mean that one model is better or worse than other. The different results are obtained from the two models depend on the definition of each norm. Each organization or company can use each model according to needs and goals.

## 6. Conclusions

The existing DEA models for measuring the relative efficiencies of a set of DMUs using various inputs to produce various outputs are limited to crisp data. But in more general cases, the data for evaluation are often collected from investigation to decide the natural language such as good, medium, and bad rather than a specific case. That is, the inputs and outputs are fuzzy. Therefore, some papers were presented the theoretical development of this technique whit fuzzy data. In this paper we initially introduced the Balf et al. [13] approach for ranking of DMUs with crisp data in DEA by using tchebycheff norm; since, inputs and outputs are fuzzy. Therefore, it may be obtained a fuzzy rank for decision making units hence we developed this method for ranking DMUs with fuzzy data. We illustrated our model by a numerical example and compared with fuzzy $L_1$ norm proposed by Jahanshahloo et al. [9]. We proposed our model for triangular fuzzy data but this model can develop for other fuzzy data simply.

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