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CAS Wavelet Function Method for Solving Abel Equations with Error Analysis

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A B S T R A C T

In this paper we use a computational method based on CAS wavelets for solving nonlinear fractional order Volterra integral equations. We solve particularly Abel equations. An operational matrix of fractional order integration for CAS wavelets is used. Block Pulse Functions (BPFs) and collocation method are employed to derive a general procedure for forming this matrix. The error analysis of proposed numerical scheme is studied theoretically. Finally, comparison of numerical results with exact solution are shown.

Keywords: Abel integral equations, CAS wavelet, fractional order volterra integral equations, operational matrix, error analysis.

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1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. It is also known as generalized integral and differential calculus of arbitrary order. Fractional differential equations are generalized from classical integer-order ones, which are obtained by replacing integer-order derivatives by fractional ones. In recent years, fractional calculus and differential equations have found enormous applications in mathematics, physics, chemistry, and engineering because of the fact that a realistic modeling of a physical phenomenon having dependence not only at the time instant but also on the previous time history can be successfully achieved by using fractional calculus. Many authors have demonstrated the applications of the fractional calculus. For examples, it has been applied to model the nonlinear oscillation of earthquakes, fluid dynamic traffic, frequency dependent damping behavior of many viscoelastic materials, continuum and statistical mechanics, colored noise, solid mechanics, economics, signal processing, and control theory [1-5]. A large class of dynamical systems appearing throughout the field of engineering and applied mathematics is described by fractional differential equations. For reason, it is indeed required a reliable and efficient techniques for the

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solution of fractional differential equations. The most frequently used methods are Walsh functions [6], Laguerre polynomials [7], Fourier series [8], Laplace transform method [9],the Jacobi polynomials [10], the Haar wavelets [11-13], Legendre wavelets [14-16], Euler wavelet [17], and the Chebyshev wavelets [18-20] have been developed to solve the fractional differential equations. Kronecker operational matrices have been developed by Kilicman for some applications of fractional calculus [21]. The operational matrix of fractional and integer derivatives has been determined for some types of orthogonal polynomials such as flatlet oblique multiwavelets [22, 23], B-spline cardinal functions [24], Legendre polynomials, Chebyshev polynomials, and CAS wavelets [28, 29]. Furthermore, the CAS wavelets have been used to approximate the solution of Volterra integral equations of the second kind [30], integro-differential equations [31], and optimal control systems by time-dependent coefficients [32, 33].

The structure of the paper is as follows. In Section 2, we introduce some basic mathematical prelimineries that we need to construct our method. We recall the basic definitions from block pulse functions and fractional calculus. In Section 3, we recall definition of CAS wavelet. The main purpose of this article is to use of an operational matix of fractional integration to reduce the solution of a fractional order Volterra integral equations to the solution of a nonlinear algebraic equations by using CAS wavelets. Then, in Section 5, we discuss on the convergence of the CAS wavelets and the error analysis for the presented method, and in Section 6 we apply our method to solving Abel integral equations. Finally, a conclusion of numerical results is presented.

2. Prelimineries

In this section, we recall the basic definitions from fractional calculus and some properties of integral calculus which we shall apply to formulate our approach.

The Riemann-Liouville fractional integral operator I^{α} of order $\alpha \geq 0$ on the usual Lebesgue space $L^1[0,b]$ is given by [34]:

$$(I^{\alpha}u)(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}u(s)ds & \alpha>0,\\ u(x) & \alpha=0.\end{cases} \tag{1}$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as:

$$D^{\alpha}u(x)=(\frac{d}{dx})^m I^{m-\alpha}u(x), \quad (m-1<\alpha\leq m), \tag{2}$$

where m is an integer.

The fractional derivative of order $\alpha > 0$ in the Caputo sense is given by [34]:

$$D_*^{\alpha}u(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1}u^{(m)}(s)ds, \quad (m-1<\alpha\le m), \tag{3}$$

where m is an integer, $\alpha>0$, and $u^{(m)}\in L^1[0,b]$.The useful relation between the Riemann-Liouville operator and Caputo operator is given by the following expression:

$$I^{\alpha}D_*^{\alpha}u(x)=u(x)-\sum_{k=0}^{m-1}u^{(k)}(0^+)\frac{x^k}{k!}, \qquad (m-1<\alpha\le m), \tag{4}$$

where m is an integer, $\alpha>0$, and $u^{(k)}\in L^1[0,b]$.

An m-set of Block Pulse Functions (BPFs) in the region of [0,T) is defined as follows:

$$b_i(t)=\begin{cases}1, & ih\le t<(i+1)h,\\ 0, & O.W,\end{cases} \tag{5}$$

where $i=1,2,...,m-1$ with positive integer values for m, and $h=\frac{T}{m}$, and m are arbitrary positive integers. There are some properties for BPFs, e.g., disjointness, orthogonality, and completeness.

The set of BPFs may be written as a m-vector B(t):

$$B(t)=[b_0(t),...,b_{m-1}(t)]^T, \tag{6}$$

where $t\in[0,1)$.

A function $f(t)\in L^2([0,1))$ may be expanded by the BPFs as:

$$f(t)\simeq\sum_{i=0}^{m-1}f_ib_i(t)=F^TB(t)=B^T(t)F, \tag{7}$$

where B(t) is given by (6) and F is a m-vector given by:

$$F=[f_0,...,f_{m-1}]^T, \tag{8}$$

the block-pulse coefficients f_i are obtained as:

$$f_i=\frac{1}{h}\int_{ih}^{(i+1)h}f(t)dt. \tag{9}$$

The integration of the vector B(t) defined in (6) may be obtained as:

$$\int_0^t B(\tau)d\tau;\Upsilon B(t), \tag{10}$$

Where Υ is called operational matrix of integration which can be represented by:

$$\Upsilon=\frac{h}{2}\begin{pmatrix}1 & 2 & 2 & \cdots & 2\\ 0 & 1 & 2 & \cdots & 2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix} \tag{11}$$

Kilicman and Al Zhour (see [35]) have given the Block Pulse operational matrix of fractional integration F^{α} as follows:

$$(I^{\alpha}B_m)(t)\simeq F^{\alpha}B_m(t) \tag{12}$$

where,

$$F^{\alpha}=\frac{1}{m}\frac{1}{\Gamma(\alpha+2)}\begin{pmatrix}1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1}\\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2}\\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3}\\ \vdots & \vdots & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & 1 & \xi_1\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}, \tag{13}$$

and $\xi_k=(k+1)^{\alpha+1}-2k^{\alpha+1}+(k-1).^{\alpha+1}$

3. CAS Wavelets

In this section, first we give some necessary definitions and mathematical preliminaries of CAS wavelets. Then function approximation via CAS wavelets and block pulse functions is introduced.

Wavelets consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [25, 26]:

$$\psi_{a,b}(t)=|a|^{\frac{-1}{2}}\psi\left(\frac{t-b}{a}\right), a,b\in \mathrm{R}, a\neq 0 \tag{14}$$

If we restrict the parameters a and b to discrete values, $a=a_0^{-k}, b=nb_0a_0^{-k}, a_0>1, b_0>0,$ where n and k are positive integers, then we have the following family of discrete wavelets:

$$\psi_{k,n}(t)=|a|^{\frac{k}{2}}\psi(a_0^k t-nb_0) \tag{15}$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0=2, b_0=1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [25, 27]. The CAS wavelets, $\psi_{nm}(t)=\psi_{k,n,m,t}$, have four arguments; $n=1,2,\ldots,2^k, k$ is any nonnegative integer, m is any integer and t is the normalized time. The orthonormal *CAS* wavelets are defined on the interval [0,1) by [27, 28]:

$$\psi_{n,m}(t)=\begin{cases}2^{k/2}CAS_m(2^k t-n), & \frac{n}{2^k}\le t<\frac{n+1}{2^k},\\ 0, & O.W.\end{cases} \tag{16}$$

where,

$$CAS_m(t)=\cos(2m\pi t)+\sin(2m\pi t), \tag{17}$$

and $n=0,1,...,2^{k-1}, k\in\mathbb{N}\cup\{0\}, m\in\mathbb{Z}.$

It is clear that CAS wavelets have compact support *i.e.*

$$Supp(\psi_{n,m}(t))=\overline{\{t:\psi_{n,m}(t)\neq 0\}}=\left[\frac{n}{2^k},\frac{n+1}{2^k}\right].$$

We introduce the following useful notation, corresponding to CAS wavelets as follows:

$$\tilde{\psi}_{n,m}(t)=\begin{cases}2^{k/2}CAS_m(n-2^k t), & \frac{n}{2^k}\le t<\frac{n+1}{2^k},\\ 0, & O.W.\end{cases} \tag{18}$$

Hint. For *m=0*, the *CAS* wavelets have the following form:

$$\psi_{n,0}(t)=2^{k/2}B_n(t)=2^{k/2}\begin{cases}1, & \frac{n}{2^k}\le t<\frac{n+1}{2^k},\\ 0, & O.W.\end{cases} \tag{19}$$

Where, $\{B_n(t)\}_{n=1}^{2^k}$ are a basis set that are called the Block Pulse Functions (BPFs) over the interval [0,1).

3.1. Function Approximation with CAS Wavelets

The set of *CAS* wavelets forms an orthonormal basis for $L^2([0,1))$. This implies that any function $f(x)$ defined over $[0,1)$ can be expanded as:

$$f(t)=\sum_{n=0}^{\infty}\sum_{m\in Z}c_{n,m}\psi_{n,m}(t) \tag{20}$$

$$\cong\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}c_{n,m}\psi_{n,m}(t)=C^T\Psi(t),$$

where $c_{n,m}=<f(t),\psi_{n,m}(t)>=\int_0^1 f(t)\psi_{n,m}(t)dt$, and $<f,g>$ is the inner product of the function f and g, C and Ψ are $2^k(2M+1)\times 1$ vectors given by:

$$C=[c_{0,-M},c_{0,-M+1},\ldots,c_{0,M},c_{1,-M},c_{1,-M+1},\ldots,c_{1,M},\ldots,c_{2^k-1,-M},\ldots,c_{2^k-1,M}]^T. \tag{21}$$

$$\Psi=[\psi_{0,-M},\psi_{0,-M+1},\ldots,\psi_{0,M},\psi_{1,-M},\psi_{1,-M+1},\ldots,\psi_{1,M},\ldots,\psi_{2^k-1,-M},\ldots,\psi_{2^k-1,M}]^T.$$

Notation. From now we define $m'=2^k(2M+1)$, such that $k,M\in\mathbb{N}\cup\{0\}$.

3.2. Operational Matrix of Fractional Integration with Hybrid Function

Eq. (7) implies that CAS wavelets can be also expanded into an m'-term BPFs as:

$$\psi_{nm}(t)\cong\sum_{i=0}^{m'-1}f_i b_i(t). \tag{22}$$

By using the properties of CAS wavelets and Eq. (9) we have:

$$f_i=m'\int_{i/m'}^{(i+1)/m'}\psi_{nm}(t)dt=\frac{m'2^{k/2}}{2^{k/2+1}m\pi}\{\sin(2m\pi(2^k\frac{i+1}{m'}-n))-\sin(2m\pi(2^k\frac{i}{m'}-n)) \tag{23}$$

$$-\cos(2m\pi(2^k\frac{i+1}{m'}-n))+\cos(2m\pi(2^k\frac{i}{m'}-n))\}$$
$$=\frac{m'}{2^{k/2+1}m\pi}\{\tilde{\psi}_{nm}(\frac{i}{m'})-\tilde{\psi}_{nm}(\frac{i+1}{m'})\},$$

for $i=n(2M+1),\ldots,(n+1)(2M+1)-1$ and otherwise $f_i=0$. Therefor we get

$$\psi_{nm}(t)\cong\frac{m'2^{k/2}}{2^{k/2+1}m\pi}[\underbrace{0,0,\ldots,0}_{n(2M+1)},\tilde{\psi}_{nm}(\frac{i}{m'})-\tilde{\psi}_{nm}(\frac{i+1}{m'}),\ldots,\tilde{\psi}_{nm}(\frac{i+2M}{m'})-\tilde{\psi}_{nm}(\frac{i+2M+1}{m'}),$$
$$0,\ldots,0]B_{m'}(t).$$

Where, $i=n(2M+1), n=0,1,\ldots,2^k-1$ and $m=-M,\ldots,M$. Therefore:

$$\Psi_{m'}(t)=\Phi_{m'\times m'}B_{m'}(t), \tag{24}$$

where $\Phi_{m'\times m'}=Diag(\Phi_0,\Phi_1,\ldots,\Phi_{2^k-1})$, and Φ_n for $n=0,1,\ldots,2^k-1$ is a $(2M+1)\times(2M+1)$ matrix which is introduces as:

$$\Phi_n=\Lambda.*\begin{bmatrix}\tilde{\psi}_{n,-M}(\frac{i}{m'})-\tilde{\psi}_{n,-M}(\frac{i+1}{m'}) & \cdots & \tilde{\psi}_{n,-M}(\frac{i+2M}{m'})-\tilde{\psi}_{n,-M}(\frac{i+2M+1}{m'})\\ \tilde{\psi}_{n,-M+1}(\frac{i}{m'})-\tilde{\psi}_{n,-M+1}(\frac{i+1}{m'}) & \cdots & \tilde{\psi}_{n,-M+1}(\frac{i+2M}{m'})-\tilde{\psi}_{n,-M+1}(\frac{i+2M+1}{m'})\\ \vdots & \vdots & \vdots\\ \tilde{\psi}_{n,0}(\frac{i}{m'}) & \cdots & \tilde{\psi}_{n,0}(\frac{i+2M}{m'})\\ \vdots & \vdots & \vdots\\ \tilde{\psi}_{n,M}(\frac{i}{m'})-\tilde{\psi}_{n,M}(\frac{i+1}{m'}) & \cdots & \tilde{\psi}_{n,M}(\frac{i+2M}{m'})-\tilde{\psi}_{n,M}(\frac{i+2M+1}{m'})\end{bmatrix},$$

where,

$$\Lambda=\frac{m'}{2^{k/2+1}\pi}\begin{bmatrix}\frac{1}{-M} & \frac{1}{-M+1} & \cdots & \frac{2^{k/2+1}\pi}{m'} & \cdots & \frac{1}{M}\\ \frac{1}{-M} & \frac{1}{-M+1} & \cdots & \frac{2^{k/2+1}\pi}{m'} & \cdots & \frac{1}{M}\\ \vdots & \vdots & & \vdots & & \vdots\\ \frac{1}{-M} & \frac{1}{-M+1} & \cdots & \frac{2^{k/2+1}\pi}{m'} & \cdots & \frac{1}{M}\end{bmatrix}.$$

Hint. $(.*)$ is point wise product.

Notation. by using the properties of CAS wavelets we now that

$$\Phi_0=\Phi_1=\ldots=\Phi_{2^k-1}.$$

For example, for k=1, M=1, the CAS operational matrix into BPFs can be expressed as

$$\Phi_{6\times 6}=\begin{pmatrix} -0.6054 & 2.4495 & -1.3045 & 0 & 0 & 0 \\ -0.9549 & 1.4142 & 0.9549 & 0 & 0 & 0 \\ -2.2594 & -2.4495 & 0.3495 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6054 & 2.4495 & -1.3045 \\ 0 & 0 & 0 & -0.9549 & 1.4142 & 0.9549 \\ 0 & 0 & 0 & -2.2594 & -2.4495 & 0.3495 \end{pmatrix}$$

Let:

$$(I^{\alpha}\Psi_{m'})(t)\cong P^{\alpha}_{m'\times m'}\Psi_{m'}(t), \tag{25}$$

where, matrix $P^{\alpha}_{m'\times m'}$ is called the CAS wavelet operational matrix of fractional integration.

Using Eqs. (24) and (12), we have:

$$(I^{\alpha}\Psi_{m'})(t)\cong (I^{\alpha}\Phi_{m'\times m'}B_{m'})(t)=\Phi_{m'\times m'}(I^{\alpha}B_{m'})(t)\cong \Phi_{m'\times m'}F^{\alpha}B_{m'}(t). \tag{26}$$

By Eqs. (25)and (26), we get:

$$P^{\alpha}_{m'\times m'}\Psi_{m'}(t);\Phi_{m'\times m'}F^{\alpha}B_{m'}(t).$$

Therefore the CAS wavelet operational matrix of fractional integration $P^{\alpha}_{m'\times m'}$ is given by (see [28]):

$$P^{\alpha}_{m'\times m'}=\Phi_{m'\times m'}F^{\alpha}\Phi^{-1}_{m'\times m'}. \tag{27}$$

For example, for k=1, M=1, the CAS operational matrix into BPFs can be expressed as

$$P^{3/4}_{6\times 6}=\begin{pmatrix} -0.0982 & 0.2634 & 0.2210 & 0.0548 & 0.0710 & 0.0714 \\ -0.1549 & 0.0182 & 0.2955 & 0.3112 & 0.2576 & 0.2320 \\ -0.3364 & -0.8969 & -0.8923 & -0.7307 & -0.6750 & -0.6346 \\ 0 & 0 & 0 & -0.0982 & 0.2634 & 0.2210 \\ 0 & 0 & 0 & -0.1549 & 0.0182 & 0.2955 \\ 0 & 0 & 0 & -0.3664 & -0.8969 & -0.8923 \end{pmatrix}.$$

4. Implementation of the Method

Consider the generalized Abel integral equation of the first and second kinds, respectively as [36]:

$$f(x)=\int_0^x \frac{y(t)}{(x-t)^{1-\alpha}}dt, \qquad 0<\alpha<1, \tag{28}$$

and,

$$y(x)=f(x)+\int_0^x \frac{y(t)}{(x-t)^{1-\alpha}}dt, \qquad 0<\alpha<1, \tag{29}$$

where $f(x)$ and $y(x)$ are differentiable functions. Here, we consider Abel integral equation as a fractional integral equation, and we use fractional calculus properties for solving these singular integral equations.

By using (1), one can write:

$$\int_0^x \frac{y(t)}{(x-t)^{1-\alpha}}dt = \Gamma(\alpha)I^{\alpha}(y(x)). \tag{30}$$

By replacing (30) in equations (28) and (29), we have:

$$f(x) = \Gamma(\alpha)I^{\alpha}(y(x)), \tag{31}$$

$$y(x) = f(x) + \Gamma(\alpha)I^{\alpha}(y(x)). \tag{32}$$

Let:

$$y(x) = U^T\Psi(x), \tag{33}$$

by substituting (33) in equations (31) and (32), we obtain:

$$f(x) = \Gamma(\alpha)I^{\alpha}(U^T\Psi(x)), \tag{34}$$

$$U^T\Psi(x) = f(x) + \Gamma(\alpha)I^{\alpha}U^T\Psi(x)). \tag{35}$$

Finally, by using equations (34) and (35), we obtain the fractional form of Abel integral equation of the first and second kind, respectively as follows:

$$f(x) = \Gamma(\alpha)U^T P^{\alpha}\Psi(x), \tag{36}$$

$$f(x) = (U^T - \Gamma(\alpha)U^T P^{\alpha})\Psi(x). \tag{37}$$

Now by collocating the equations (36) and (37) at collocation points $\{x_i\}_{i=1}^{2^k(2M+1)}$ where $x_i = \dfrac{i-\frac{1}{2}}{2^k(2M+1)}$ are the CAS wavelet points of degree $m' = 2^k(2M+1)$, we obtain the following system of algebraic equations:

$$f(x_i) = \Gamma(\alpha)U^T P^{\alpha}\Psi(x_i), \tag{38}$$

$$f(x_i) = (U^T - \Gamma(\alpha)U^T P^{\alpha})\Psi(x_i). \tag{39}$$

Clearly, by solving this system and determining $U = [u_1, u_2, ..., u_{2^k(2M+1)}]^t$, we obtain the approximate solution of the equations (36) and (37) as $y(x) = U^T\Psi(x)$.

5. Error Analysis

In this section, we provide a theoretical error and convergence analysis of the proposed method for solving Abel integral equations. At first, we indicate that the CAS wavelet expansion of a function $f(x)$, with bounded second derivative, converges uniformly to $f(x)$. But before that, for ease reference, we present the following theorem:

Theorem 5.1. If the CAS wavelet expansion of a continuous function $f(x)$ converges uniformly, then the CAS wavelet expansion converges to the function $f(x)$.

Proof. Let:

$$g(x)=\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}}c_{n,m}\psi_{n,m}(x), \tag{40}$$

where $c_{n,m}=<f(t),\psi_{n,m}(t)>$. Multiplying both sides of (40) by $\psi_{n,m}(x)$, in which p and q are fixed and then integrating term wise, justified by uniformly convergence, on [0,1], we have:

$$\int_0^1 g(x)\psi_{pq}(x)dx=\int_0^1\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}}c_{n,m}\psi_{n,m}(x)\psi_{pq}(x)dx. \tag{41}$$

$$=\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}}c_{nm}\int_0^1\psi_{nm}(x)\psi_{pq}(x)dx=c_{pq}. \tag{42}$$

Thus $<g(x),\psi_{nm}(x)>=c_{nm}$. for $n\in\mathbb{N}$ and $m\in\mathbb{Z}$. Consequently f,g have same Fourier expansions with the CAS wavelet basis and therefore $f(x)=g(x);(0\le x\le 1)$ [37].

Theorem 5.2 (see [38]). A function $f(x)\in L^2[0,1]$, with bounded second derivative, say $|f''(x)|\le\gamma$, can be expanded as an infinite sum of the CAS wavelets and the series converges uniformly to $f(x)$ that is

$$f(x)=\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}}c_{nm}\psi_{nm}(x)$$

Furtheremore, we have:

$$P_{k,M}f-f_{\infty}\le\frac{\gamma}{\pi^2}\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{\frac{5}{2}}m^2}, x\in[0,1]. \tag{43}$$

Proof. From Eq.(20), it follows that :

$$c_{nm}=<f(x),\psi_{nm}(x)>=\int_0^1 f(x)\psi_{nm}(x)dx=\int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}}2^{\frac{k}{2}}f(x)CAS_m(2^kx-n+1)dx. \tag{44}$$

By substituting $2^kx-n+1=t$ in Eq.(44), yields:

$$c_{nm}=\frac{1}{2^{\frac{k}{2}}}\int_0^1 f(\frac{t+n-1}{2^k})CAS_m(t)dt \tag{45}$$

$$=\frac{1}{2^{\frac{k}{2}}}\int_0^1 f(\frac{t+n-1}{2^k})d(\frac{sin(2m\pi t)-cos(2m\pi t)}{2m\pi}) \tag{46}$$

$$=\frac{1}{2^{\frac{k}{2}}}f(\frac{t+n-1}{2^k})d(\frac{sin(2m\pi t)-cos(2m\pi t)}{2m\pi})|_0^1-\frac{1}{2^{\frac{3k}{2}}(2m\pi)}\int_0^1 f'(\frac{t+n-1}{2^k})((sin(2m\pi t)-cos(2m\pi t))dt \tag{47}$$

$$=\frac{1}{2^{\frac{3k}{2}}(2m\pi)}\int_0^1 f'(\frac{t+n-1}{2^k})d(\frac{CAS_m(t)}{2m\pi}) \tag{48}$$

$$=\frac{1}{2^{\frac{3k}{2}}(2m\pi)^2}f'(\frac{t+n-1}{2^k})CAS_m(t)|_0^1-\frac{1}{2^{\frac{5k}{2}}(2m\pi)^2}\int_0^1 f''(\frac{t+n-1}{2^k})(CAS_m(t))dt \tag{49}$$

Thus, we get:

$$|c_{nm}|^2 = \left|\frac{1}{2^{\frac{5k}{2}}(2m\pi)^2}\int_0^1 f''\left(\frac{t+n-1}{2^k}\right)(CAS_m(t))dt\right|^2 \tag{50}$$

$$\leq \left(\frac{1}{2^{\frac{5k}{2}}(2m\pi)^2}\right)^2 \int_0^1 \left|f''\left(\frac{t+n-1}{2^k}\right)\right|^2 dt \int_0^1 |CAS_m(t)|^2\ dt \tag{51}$$

$$\leq \left(\frac{N}{2^{\frac{5k}{2}}(2m\pi)^2}\right)^2 \int_0^1 |CAS_m(t)|^2\ dt \tag{52}$$

from orthonormality of CAS wavelets, we know that $\int_0^1 |CAS_m(t)|^2\ dt = 1$. Since $n \leq 2^k$, we have:

$$|c_{nm}| \leq \frac{\gamma}{4\pi^2 n^{\frac{5}{2}} m^2}. \tag{53}$$

Hence, the series $\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}$ is absolutely convergent. On the other hand, we have:

$$\left|\sum_{n=0}^{\infty}\sum_{m\in Z} c_{nm}\psi_{nm}(x)\right| \leq \sum_{n=0}^{\infty}\sum_{m\in Z} |c_{nm}|\,|\psi_{nm}(x)| \leq 2\sum_{n=0}^{\infty}\sum_{m\in Z} |c_{nm}| \leq \infty \tag{54}$$

Accordingly, utilizing Theorem (5.1), the series $\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(x)$ converges to $f(x)$ uniformly. Moreover, we conclude that:

$$P_{k,M} f(x) - f(x)_{\infty} = 2\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty} c_{nm}\psi_{nm}(x)_{\infty} \leq 4\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty} |c_{nm}| \tag{55}$$

Now, from (53), we obtain:

$$P_{k,M} f(x) - f(x)_{\infty} \leq \frac{\gamma}{\pi^2}\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{\frac{5}{2}} m^2} \tag{56}$$

This completes the proof.

Now, we proceed by discussing the convergence of the presented method.

Theorem 5.3. Suppose that $f(x) \in L^2[0,1]$, with bounded second derivative, $|f''(x)| \leq \gamma$, and $P_{k,M} f(x)$ the truncated expantion of CAS wavelet for $f(x)$ by (20). Then we have the error bound as follows:

$$\| e_{k,m}(x) \|_E = \| P_{k,M} f(x) - f(x)_{\infty} \|_E \leq \left\| \frac{\gamma}{\pi^2}\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{\frac{5}{2}} m^2} \right\|, \tag{57}$$

where,

$$\| e_{k,m}(x) \|_E = \int_0^1 (e_{k,m}(x))^2 dx \cong \int_0^1 \frac{\gamma}{\pi^2}\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{\frac{5}{2}} m^2}. \tag{58}$$

Proof. Proof easily results from theorem (5.2).

Now, by increasing $m' = 2^k(2M+1)$ the error function, $e_{k,m}(x)$, approaches to zero. If $e_{k,m}(x) \to 0$ when m' is sufficiently large enough, then the error decreases.

6. Numerical Examples

To show the efficiency of the proposed method, we will apply our method to obtain the approximate solution of the following examples. All of the computations have been performed using MATLAB 7.8.

Example 1. Consider the first kind Abel integral equation of the form:

$$x = \int_0^x \frac{y(t)}{(x-t)^{\frac{4}{5}}} dt. \tag{59}$$

The exact solution is $y(x) = \frac{5}{4}\frac{\sin(\frac{\pi}{5})}{\pi} x^{\frac{4}{5}}$.

Let $y(x) = U^T \Psi(x)$ and $X = \frac{l-.5}{m^{'}}$, for $l = 1,2,...,m^{'}$, are collocation points.

Now from Equations (1), (24) and (25) we have:

$$\int_0^x \frac{y(t)}{(x-t)^{\frac{4}{5}}} dt = \Gamma(\frac{1}{5}) I^{\alpha}(y(x)) \tag{60}$$

$$= \Gamma(\frac{1}{5}) I^{\alpha}(U^T \Psi(x))$$

$$= \Gamma(\frac{1}{5}) U^T P^{\alpha} \Phi_{m^{'} \times m^{'}} B(x).$$

So we obtain algebraic equations form of Example 1 as follows:

$$X^T - \Gamma(\frac{1}{5}) U^T P^{\alpha} \Phi_{m^{'} \times m^{'}} = 0. \tag{63}$$

By solving this system and determining U, we obtain the approximate solution of equation (59) as $y(x) = U^T \Psi(x)$.

Plot of error that is resulted by method, for $m^{'} = 12$, is illustrated in Figure 1.

Example 2. Consider the second kind Abel integral equation of the form:

$$y(x) = x^2 + \frac{16}{15} x^{\frac{5}{2}} - \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}} dt. \tag{64}$$

The exact solution is $y(x) = x^2$.

Let $y(x) = U^T \Psi(x)$ and $X = \frac{l-.5}{m^{'}}$, for $l = 1,2,...,m^{'}$. By Equations (1), (24) and (25) we have:

$$\int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}} dt = \Gamma(\frac{1}{2}) I^{\alpha}(y(x)) \tag{65}$$

$$= \Gamma(\frac{1}{2}) I^{\alpha}(U^T \Psi(x)) \tag{66}$$

$$= \Gamma(\frac{1}{2})U^T P^{\alpha} \Phi_{m^{'} \times m^{'}} B(x). \tag{67}$$

Then we obtain algebraic equations form of example 2 as follows:

$$U^T \Phi_{m^{'} \times m^{'}} - (X^2)^T - \frac{16}{15}(X^{\frac{5}{2}})^T + \Gamma(\frac{1}{2})U^T P^{\alpha} \Phi_{m^{'} \times m^{'}} = 0. \tag{68}$$

Figure 2 shows error Plot for Example 2, using presented method with $m^{'} = 12$.

Example 3. Now consider another second kind Abel integral equation of the form:

$$y(x) = 2\sqrt{x} - \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}} dt. \tag{69}$$

The exact solution is $y(x) = 1 - e^{\pi x} erfc(\sqrt{\pi x})$.

Where,

$$erfc(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{70}$$

Figure 3 shows Error plot for Example 3 with $m^{'} = 18$.

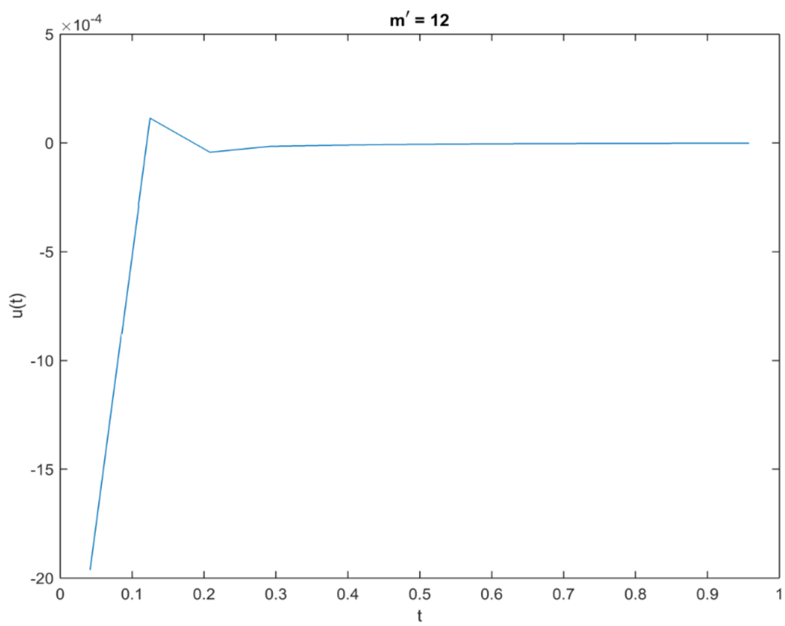

Figure 1. Error plot for Example 1 with $m^{'} = 12$.

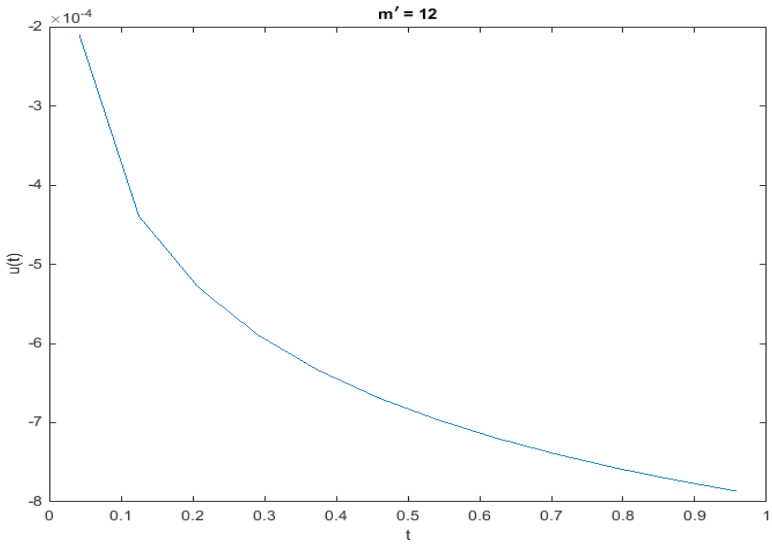

Figure 2. Error plot for Example 2 with $m' = 12$.

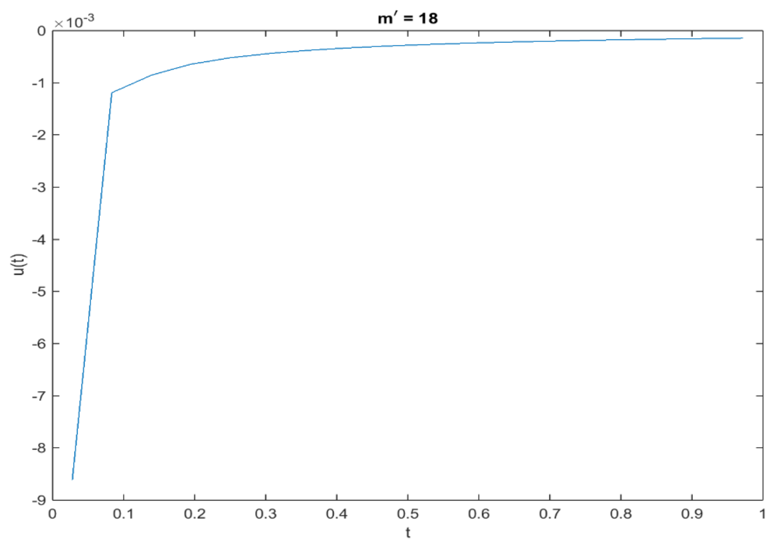

Figure 3. Error plot for Example 3 with $m' = 18$.

7. Conclusions

In this paper, we have presented a numerical scheme for solving Abel integral equations of the first and second kinds. The method which is employed is based on the CAS wavelet. By considering Abel integral equations of the first and second kinds as a fractional integral equation, we use fractional calculus properties for solving these singular integral equations. Error analysis is provided for the new method. Figure 1-3 show the error of presented method and the exact solution of the Abel equations proposed in examples 1-3 respectively. The obtained results shows that the used technique can solve the fractional order Volterra integral equations effectively.

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