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# The Survey of Data Envelopment Analysis Models in Fuzzy Stochastic Environments

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**A B S T R A C T**

One of the best techniques for evaluating the performance of organizations is data envelopment analysis. Data Envelopment Analysis (DEA) is a non-parametric method for evaluating the performance of Decision-Making Units (DMUs) that recognizes the relative performance of DMUs based on mathematical programming. The classic DEA model were initially formulated for optimal inputs and outputs, but in real-world problems the values observed from input and output data are often ambiguous and random. In fact, decision makers may be faced with a specific hybrid environment where there is fuzziness and randomness in the problem. To overcome this problem, data envelopment analysis models in random fuzzy environment have been proposed. Although the DEA has many advantages, one of the disadvantages of this method is that the classic DEA does not actually give us a definitive conclusion and does not allow random changes in input and output. In this research data envelopment analysis models in fuzzy random environments is reviewed.

**Keywords:** DEA, Decision Making Unit, Performance, Random Fuzzy Data Envelopment Analysis.

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## 1. Introduction

Data Envelopment Analysis (DEA) is a method based on linear programming models to measure the relative performance of Decision-Making Units (DMUs) congruent with multiple inputs and outputs. The general view in unit evaluation is that reducing inputs and increasing outputs improves the performance and best performance of data envelopment analysis models . Classic DEA models were initially formulated only for optimal inputs and outputs. In the real world, the values observed from input and output data are often significant and random. In fact, decision makers may encounter a specific hybrid environment in which fuzziness and

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randomness exist simultaneously. However, undesirable outputs may also be present in the production process, which should be minimized. Consider, for example, a paper mill that produces undesirable environmental outputs such as sulfur oxides and suspended particulates in the paper production process as desirable outputs. If there are inefficiencies along the way, the efficiency needs to be reduced to improve efficiency, which means that when evaluating the performance of how to deal with undesirable outputs and desirable outputs, it should be different.

To deal with imprecise data, the notions of fuzziness and randomness were introduced in DEA. Fuzzy sets can be used to represent ambiguous or imprecise information. On the other hand, the data can be obtained by statistics in some measurement errors and data entry errors characterized by random variables. However, in many practical situations, there is not a sufficient number of crisp statistic data. To handle such circumstances, a twofold uncertainty is needed.

Hatami-Marbini et al. [1] classified the fuzzy DEA methods in the literature into five general groups: Tolerance approach [2, 3], α-level based approach, fuzzy ranking approach [4, 5], possibility approach [6], and fuzzy arithmetic approach [7]. Among these approaches, the α-level based approach is probably the most popular fuzzy DEA model in the literature. This approach generally tries to transform the FDEA model into a pair of parametric programs for each α-level. Kao and Liu [8], one of the most cited studies in the α-level approach's category, used Chen and Klein [9] method for ranking fuzzy numbers to convert the FDEA model to a pair of parametric mathematical programs for the given level of α. Saati et al. [10] proposed a fuzzy CCR model as a possibilistic programming problem and changed it into an interval programming problem by means of the α-level based approach. Parameshwaran et al. [11] proposed an integrated fuzzy analytic hierarchy process and DEA approach for the service performance measurement. Puri and Yadav [12] applied the suggested methodology by Saati et al. [10] to solve fuzzy DEA model with undesirable outputs. Khanjani et al. [13] proposed the fuzzy free disposal hull models under possibility and credibility measures. Momeni et al. [14] used fuzzy DEA models to address the impreciseness and ambiguity associated with input and output data in supply chain performance evaluation problems. Payan [15] evaluated the performance of DMUs with fuzzy data by using the common set of weights based on a linear program. Aghayi et al. [16] formulated a model to measure the efficiency of DMUs with interval inputs and outputs based on common sets weights.

In order to evaluate the efficiency of DMUs with the deterministic inputs and the random outputs, Land et al. [17] extended the chance constrained DEA model. Olesen and Petersen [18] developed the chance constrained programming model for efficiency evaluation using a piecewise linear envelopment of confidence region for observed stochastic multiple-input multiple-output combinations in DEA. Huang and Li [19] developed stochastic models in DEA by taking into account the possibility of random variations in input-output data. Cooper et al. [20], Li [21], and Bruni et al. [22] utilized the joint chance constraints to extend the concept of stochastic efficiency. Cooper et al. [23] used chance-constrained programming for extending congestion DEA models. Tsionas and Papadakis [24] developed Bayesian inference techniques in chance-constrained DEA models. Udhayakumar et al. [25] used a genetic algorithm to solve the chance-constrained DEA models involving the concept of satisficing. Also some of the banking applications in relation to satisficing DEA can be found in Udhayakumar et al. [25] and Tsolas and Charles [26]. Franoosh et al. [27] proposed chance-constrained FDH model with random input and random output. Wu et al. [28] proposed a stochastic DEA model by considering undesirable outputs with weak disposability. This model not only deals with the existence of random errors in the collected data, but also depicts the production rules uncovered by weak disposability of the undesirable outputs. Also, a comparison work between stochastic DEA and fuzzy DEA approaches have been introduced to evaluate the efficiency of Angolan banks by Wanke et al. [29]. A review of stochastic DEA models can be found in a recent work by Olesen and Petersen [30].

However, in the real-world problem decision makers may need to base decisions on information which are both fuzzily imprecise and probabilistically uncertain. Kwakernaak [31] introduced the concept of fuzzy random variable, and then this idea enhanced by a number of researchers in the literature [32, 33, 34, 35]. Qin and Liu [35] developed a Fuzzy Random DEA (FRDEA) model where randomness and fuzziness exist simultaneously [35]. The authors characterized the fuzzy random data with known possibility and probability distributions. Tavana et al. [36] also introduced three different FDEA models consisting of probability-possibility, probability-necessity and probability-credibility constraints in which input and output data entailed fuzziness and randomness at the same time. Also, Tavana et al. [37] provided a chance-constrained DEA model with random fuzzy inputs and outputs with poisson, uniform and normal distributions. After that, Tavana et al. [38] proposed DEA models with birandom input-output. Khanjani et al. [13] proposed fuzzy rough DEA models based on the expected value and possibility approaches. Paryab et al. [39] proposed DEA models using bi-fuzzy data based possibility approach. However, there has been no attempt to study randomness and roughness simultaneously in DEA problems. Tavana et al. [40] also introduced a DEA model for problems characterized by random-rough variables. Nasseri et al. [41] proposed a new approach to consider the impact of undesirable output on the performance of DMUs in fuzzy stochastic environment. To deal with the uncertain environments, especially hybrid environments, the DEA model may disorder its structure when the uncertain parameter of input and output exist. For example, the method proposed by Tavana et al. [36] does not compute the efficiency scores of DMUs in the range of zero to one for input-oriented DEA models.

Classical DEA generally uses deterministic data to evaluate performance, but today in the real world, uncertainties in data are clearly evident, with few studies available. This paper reviewed the most important models of DEA with fuzzy stochastic environments.

## 2. Preliminaries

### 2.1. Definition and Basic Concept

In this subsection, we review some necessary concepts related to this research, which will be used in the rest of paper.

**Definition 1.** A fuzzy set $\widetilde{A}$, defined on universal set X, is given by a set of ordered pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | x \in X\}$ where $\mu_{\widetilde{A}}(x)$ gives the membership grade of the element $x$in the set $\tilde{A}$and is called membership function.

**Definition 2.** A fuzzy set $\widetilde{A} = (m, \alpha, \beta)_{LR}$t is said to be an LR fuzzy number, if its membership function is given by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\dfrac{m-x}{\alpha}), \text{for} x \leq m, \alpha > 0, \\ 1, \text{for} x = m, \\ R(\dfrac{x-n}{\beta}), \text{for} x \geq n, \beta > 0. \end{cases}$$

Where $L: [0, \infty) \rightarrow [0,1]$ and $R: [0, \infty) \rightarrow [0,1]$ are non-increasing on such that L(0)=R(0)=1.

**Remark 1.** If $L(x) = R(x) = \max\{0, 1-x\}$ then an LR fuzzy number $\widetilde{A} = (m, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number and is denoted by $\widetilde{A} = (m, \alpha, \beta)$.

**Definition 3.** A fuzzy set $\widetilde{A}$, defined on universal set of real numbers R , is said to be a fuzzy number if its membership function has the following characteristics:

- $\widetilde{A}$ is convex, i.e. $\forall x, y \in R, \forall \lambda \in [0,1], \mu_{\widetilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)\}$.
- $\widetilde{A}$ is normal, i.e. $\exists \overline{x} \in R; \mu_{\widetilde{A}}(\overline{x}) = 1$.
- $\mu_{\widetilde{A}}$ is piecewise continuous.

**Definition 4.** Let $\widetilde{A} = (m, \alpha, \beta)_{LR}$ be an LR fuzzy number and $\lambda$ be a real number in the interval [0,1] then the crisp set, $\widetilde{A}_{\lambda} = \{x \in R: \mu_{\widetilde{A}}(x) \geq \lambda\} = [m - \alpha L^{-1}(\lambda), m + \beta R^{-1}(\lambda)]$ is said to be $\lambda -$ cut of $\widetilde{A}$.

**Definition 5.** Let $\widetilde{A}_1 = (m_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A}_2 = (m_2, \alpha_2, \beta_2)_{LR}$ be two LR fuzzy numbers and k be a non-zero real number. Then the exact formula for the extended addition and the scalar multiplication are defined as follows:

$$(m_1, \alpha_1, \beta_1)_{LR} + (m_2, \alpha_2, \beta_2)_{LR} = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}.$$
$$k > 0, k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, k\alpha_1, k\beta_1)_{LR}.$$
$$k < 0, k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, -k\beta_1, -k\alpha_1)_{LR}.$$

**Definition 6. (Extension principle).** This principle allows the generalization for crisp mathematical concepts in fuzzy frameworks. For any function f, mapping points in set X to points in set Y, and fuzzy set $A \in P(X)$where $A = \mu_1(x_1) + \mu_2(x_2) + \ldots + \mu_n(x_n)$, this principle expresses $f(A) = f(\mu_1(x_1) + \mu_2(x_2) + \ldots + \mu_n(x_n)) = f(\mu_1(x_1)) + f(\mu_2(x_2)) + \ldots + f(\mu_n(x_n))$.

**Definition 7.** Let $(\theta, P(\theta), Pos)$be a possibility space where $\theta$ is a non-empty set involving all possible events, and $P(\theta)$ is the power set of $\theta$. For every$A \in P(\theta)$ there is a non-negative number $Pos(A)$ , so-called a possibility measure, satisfying the following axioms:

- $P(\phi) = 0, P(\theta) = 1$.
- for every $A, B \in P(\theta), A \subseteq B$implies $Pos(A) \leq Pos(B)$.
- for every subset $\{A_w : w \in W\} \subseteq P(\theta), Pos(\cup_w A_w) = Sup_w Pos(A_w)$.

The elements of $P(\theta)$are also called fuzzy events.

**Definition 8.** Let $\xi$be a fuzzy variable on a possibility space$(\theta, P(\theta), Pos)$. The possibility of fuzzy event $\{\xi \geq r\}$ , where r is any real number, is defined $Pos(\xi \geq r) = Sup_{t \geq r} \mu_\xi(t)$ , where is the membership function of $\xi$.

**Definition 9.** Let $(\Omega, A, Pr)$ be a probability space where $\Omega$ is a sample space, A is the s-algebra of subsets of $\Omega$ (i.e. the set of all possible potentially interesting events), and Pr is a probability measure on $\Omega$. A Fuzzy Random Variable (FRV) is a function $\xi$ from a probability space$(\Omega, A, Pr)$to the set of fuzzy variables such that for every Borel set B of R, $Pos\{\xi(w), w \in B\}$is a measurable function of $\omega$.

**Definition 10.** A fuzzy random vector is a map from a sample space to a collection of fuzzy vectors, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) : \Omega \to F_v^n$ , such that for any closed subset $F \in R^n$, $Pos\{\gamma | \xi(\omega, \gamma) \in F\}$is a measurable function of $\omega \in \Omega$ , i.e. for any $t \in [0,1]$, we have $\{\omega \in \Omega | Pos\{\gamma | \xi(\omega, \gamma) \in F\} \leq t\} \in A$ . In the case of n=1, $\xi$is called a fuzzy random variable.

**Definition 11. (Fuzzy random arithmetic).** Let $\xi_1$and$\xi_2$be two FRVs with the probability spaces $(\Omega_1, A_1, Pr_1)$ and $(\Omega_2, A_2, Pr_2)$, respectively. Then $\xi = \xi_1 + \xi_2$ is defined as $\xi(\omega_1, \omega_2) = \xi_1(\omega_1) + \xi_2(\omega_2)$ for any $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ , where $(\Omega_1 \times \Omega_2, A_1 \times A_2, Pr_1 \times Pr_2)$ is the corresponding probability space.

**Definition 12.** Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a fuzzy random vector, and $f : \Re^n \to \Re$ be a continuous function. Then $f(\xi)$ will be a fuzzy random variable.

**Definition 13.** An LR fuzzy random variable will be denoted by $\xi(\omega)$, where $\omega \in \Omega$ and described by the following membership function:

$$\mu_{\xi(\omega)}(x) = \begin{cases} L(\dfrac{m(\omega) - x}{\alpha}), x \leq m(\omega), \\ 1 x = m(\omega), \\ R(\dfrac{x - m(\omega)}{\beta}), x \geq m(\omega). \end{cases}$$

Where $m(\omega)$ is the normally distributed random variable?

## 3. Conventional DEA-CCR Model with Crisp Data

According to the CCR model, proposed by Charnes et al. [42], under a Constant Returns to Scale (CRS) technology, we consider that there are n DMUs to be evaluated where every $DMU_j$, j=1,…,n, produces s outputs, $y_{rj}$(r=1,…,s) using m inputs, $x_{ij}$(i=1,…,m). The following problem is used to evaluate the technical radial input-efficiency of a given $DMU_p$:

$$\begin{aligned}
&\theta_p^* = \max \sum_{r=1}^{s} u_r y_{rp} \\
&\text{s.t.} \sum_{i=1}^{m} v_i x_{ip} = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \\
&u_r, v_i \ge 0, r = 1,\dots,s, i = 1,\dots,m.
\end{aligned} \tag{1}$$

Where the $u_r$ and $v_i$ are the weights assigned to the rth output and ith input, respectively. The $DMU_p$is technically efficient if $\theta_p^* = 1$, otherwise $DMU_p$is inefficient.

## 4. Existing Models

In this section, we first review the proposed models of Tavana et al. [37]. In a random environment and then a random fuzzy environment.

Tavana et al. [43] presented the mathematical details of the proposed approach for solving the CCR model in which input and output data are assumed to be the random variables. The details of this proposed model are as follows:

Let us assume $\bar{\tilde{x}}_j = (\bar{\tilde{x}}_{1j},\dots,\bar{\tilde{x}}_{mj})^T \in \Re^m$ and $\bar{\tilde{y}}_j = (\bar{\tilde{y}}_{1j},\dots,\bar{\tilde{y}}_{sj})^T \in \Re^s$are the random input and output data for $DMU_j$, $j = 1,\dots,n$, and each of them has a normal distribution. Also, let us assume $\bar{x}_j = (\bar{x}_{1j},\dots,\bar{x}_{mj})^T \in \Re^m$ and $\bar{y}_j = (\bar{y}_{1j},\dots,\bar{y}_{mj})^T \in \Re^s$are the expected vectors of the inputs and outputs of $\bar{\tilde{x}}_j$ and $\bar{\tilde{y}}_j$, respectively.

Refer to [43], the final formula will be as follows:

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \sum_{j=1}^{n} x_{ij} - x_{ip}\theta + \bar{v}_i \phi^{-1}(1-\alpha) + v_i \phi^{-1}(1-\beta) \le 0, i = 1, \dots, m, \\
&\sum_{j=1}^{n} y_{rj}\lambda_j - y_{rp} - \bar{u}_r \phi^{-1}(1-\alpha) - u_r \phi^{-1}(1-\beta) \ge 0, r = 1, \dots, s, \\
&v_i^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j \lambda_k \mathrm{Cov}(\bar{x}_{ij}, \bar{x}_{ik}) + \theta^2 \mathrm{Var}(\bar{x}_{ip}) - 2\theta \sum_{j=1}^{n} \lambda_j \mathrm{Cov}(\bar{x}_{ij}, \bar{x}_{ip}), i = 1, \dots, m, \\
&\bar{v}_i^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j \lambda_k \mathrm{Cov}(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ik}) + \theta^2 \mathrm{Var}(\tilde{\bar{x}}_{ip}) - 2\theta \sum_{j=1}^{n} \lambda_j \mathrm{Cov}(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ip}), i = 1, \dots, m, \qquad (2) \\
&\bar{u}_r^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j \lambda_k \mathrm{Cov}(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rk}) + \mathrm{Var}(\tilde{\bar{y}}_{rp}) - 2 \sum_{j=1}^{n} \lambda_j \mathrm{Cov}(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rp}), r = 1, \dots, s, \\
&u_r^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j \lambda_k \mathrm{Cov}(\bar{y}_{rj}, \bar{y}_{rk}) + \mathrm{Var}(\bar{y}_{rp}) - 2 \sum_{j=1}^{n} \lambda_j \mathrm{Cov}(\bar{y}_{rj}, \bar{y}_{rp}), r = 1, \dots, s, \\
&\lambda_j, v_i, \bar{v}_i, u_r, \bar{u}_r \ge 0, \\
&j = 1, \dots, n; i = 1, \dots, m; r = 1, \dots, s.
\end{aligned}
$$

**Definition 14.** A DMU is said to be probabilistic-probabilistic $\alpha - \beta$ efficient if the optimal value of the objective function of model (2) is equal to 1 at the probability level $\alpha$and probability level $\beta$; otherwise, it is said to be probabilistic-probabilistic $\alpha - \beta$ inefficient.

**Proposition 1.** Model (2) for any $\alpha$ and $\beta$ level is feasible.

Proof: Let $\lambda_j = \begin{cases} 1 j = p \\ 0 j \neq p \end{cases}, j = 1, \dots, n, \theta = 1$. Then $v_i = 0, \bar{v}_i = 0, u_r = 0, \bar{u}_r = 0$. This solution is a feasible solution for model (2).

Also, similar to the random CCR model proposed, the super-efficiency random model is developed to improve the discrimination power. The corresponding model is as follows:

$$
\begin{aligned}
&\min \theta^{\text{Super}} \\
&\text{s.t.} \sum_{\substack{j=1 \\ j\neq p}}^{n} x_{ij}\lambda_j - x_{ip}\theta + \overline{v}_i\phi^{-1}(1-\alpha) + v_i\phi^{-1}(1-\beta) \le 0, i = 1,\dots,m, \\
&\sum_{\substack{j=1 \\ j\neq p}}^{n} y_{rj}\lambda_j - y_{rp} - \overline{u}\phi^{-1}(1-\alpha)_r + u_r\phi^{-1}(1-\beta) \ge 0, r = 1,\dots,s, \\
&v_i^2 = \sum_{\substack{j=1 \\ j\neq p}}^{n}\sum_{\substack{k=1 \\ j\neq p}}^{n} \lambda_j\lambda_k \mathrm{Cov}(\overline{x}_{ij},\overline{x}_{ik}) + \theta^2 \mathrm{Var}(\overline{x}_{ip}) - 2\theta\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j \mathrm{Cov}(\overline{x}_{ij},\overline{x}_{ip}), i = 1,\dots,m, \\
&\overline{v}_i^2 = \sum_{\substack{j=1 \\ j\neq p}}^{n}\sum_{\substack{k=1 \\ j\neq p}}^{n} \lambda_j\lambda_k \mathrm{Cov}(\bar{\tilde{x}}_{ij},\bar{\tilde{x}}_{ik}) + \theta^2 \mathrm{Var}(\bar{\tilde{x}}_{ip}) - 2\theta\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j \mathrm{Cov}(\bar{\tilde{x}}_{ij},\bar{\tilde{x}}_{ip}), i = 1,\dots,m, \\
&\overline{u}_r^2 = \sum_{\substack{j=1 \\ j\neq p}}^{n}\sum_{\substack{k=1 \\ j\neq p}}^{n} \lambda_j\lambda_k \mathrm{Cov}(\bar{\tilde{y}}_{rj},\bar{\tilde{y}}_{rk}) + \mathrm{Var}(\bar{\tilde{y}}_{rp}) - 2\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j \mathrm{Cov}(\bar{\tilde{y}}_{rj},\bar{\tilde{y}}_{rp}), r = 1,\dots,s, \\
&u_r^2 = \sum_{\substack{j=1 \\ j\neq p}}^{n}\sum_{\substack{k=1 \\ j\neq p}}^{n} \lambda_j\lambda_k \mathrm{Cov}(\overline{y}_{rj},\overline{y}_{rk}) + \mathrm{Var}(\overline{y}_{rp}) - 2\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j \mathrm{Cov}(\overline{y}_{rj},\overline{y}_{rp}), r = 1,\dots,s, \\
&\lambda_j, v_i, \overline{v}_i, u_r, \overline{u}_r \ge 0, \\
&j = 1,\dots,n; i = 1,\dots,m; r = 1,\dots,s.
\end{aligned}
\tag{3}
$$

Now we want to illustrate two models of data envelopment analysis with fuzzy and random inputs and outputs. Tavana et al. [36] offered the mathematical details of the probability-possibility, probability-necessity and probability-credibility approaches for solving the CCR models in which the input and output data are assumed to be characterized by Fuzzy Random Variables (FRVs).

**Theorem 1.** Assume that $\xi$ is a fuzzy random vector, and $g_j$ are real-valued continuous functions for j=1,…,n. We have:

- The possibility $\mathrm{Pos}\{g_j(\xi(\omega)) \le 0, j = 1,\dots,p\}$ is a random variable.
- The necessity $\mathrm{Nec}\{g_j(\xi(\omega)) \le 0, j = 1,\dots,p\}$ is a random variable.
- The credibility $\mathrm{Cr}\{g_j(\xi(\omega)) \le 0, j = 1,\dots,p\}$ is a random variable.

They considered n DMUs, each of consumes m fuzzy stochastic inputs, denoted by $\bar{\tilde{x}}_{ij} = (x_{ij}^m, x_{ij}^\alpha, x_{ij}^\beta)$, i=1,…,m, j=1,…,n, and produces s fuzzy stochastic outputs, denoted by $\bar{\tilde{y}}_{rj} = (y_{rj}^m, y_{rj}^\alpha, y_{rj}^\beta)$, r=1,…,s, j=1,…,n. Let $x_{ij}^m$ and $y_{rj}^m$, denoted by $x_{ij}^m \sim N(x_{ij}, \sigma_{ij}^2)$ and $y_{rj}^m \sim N(y_{rj}, \sigma_{rj}^2)$ be normally distributed. Therefore, $x_{ij}(y_{rj})$ and $\sigma_{ij}^2(\sigma_{rj}^2)$ are the mean and the variance of $x_{ij}^m(y_{rj}^m)$ for $\mathrm{DMU}_j$, respectively. Finally the final probability-possibility CCR model is as follows:

$$
\begin{aligned}
&\max \varphi\\
&\text{s.t. } \varphi - \sum_{r=1}^{s} u_r y_{rp} - R^{-1}(\delta) \sum_{r=1}^{s} u_r y_{rp}^{\beta} \le \overline{\sigma}_p^{C} \phi_{1-\gamma}^{-1},\\
&\sum_{i=1}^{m} v_i x_{ip} + R^{-1}(\delta) \sum_{i=1}^{m} v_i x_{ij}^{\beta} + \overline{\sigma}_p^{I} \phi_{1-\gamma}^{-1} \ge 1,\\
&\sum_{i=1}^{m} v_i x_{ip} - L^{-1}(\delta) \sum_{i=1}^{m} v_i x_{ij}^{\alpha} - \overline{\sigma}_p^{I} \phi_{1-\gamma}^{-1} \le 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - (L^{-1}(\delta_j) \sum_{r=1}^{s} u_r y_{rp}^{\alpha} + R^{-1}(\delta_j) \sum_{i=1}^{m} v_i x_{ij}^{\beta}) - \overline{\sigma}_p^{A} \phi_{1-\gamma_j}^{-1} \le 0, j = 1,\dots,n, \qquad (4)\\
&(\overline{\sigma}_p^{C})^2 = (\sum_{r=1}^{s} u_r^2\, \sigma_{rp}^2),\\
&(\overline{\sigma}_p^{I})^2 = (\sum_{i=1}^{m} v_i^2\, \sigma_{ip}^2),\\
&(\overline{\sigma}_p^{A})^2 = (\sum_{r=1}^{s} u_r^2\, \sigma_{rj}^2 + \sum_{i=1}^{m} v_i^2\, \sigma_{ij}^2), j = 1,\dots,n,\\
&u_r, v_i, \overline{\sigma}_p^{C}, \overline{\sigma}_p^{I}, \overline{\sigma}_p^{A} \ge 0, r = 1,\dots,s, i = 1,\dots,m, j = 1,\dots,n.
\end{aligned}
$$

They presented the probability-necessity CCR model and it is as follows:

$$
\begin{aligned}
&\max \overline{\varphi}\\
&\text{s.t.} \sum_{r=1}^{s} u_r y_{rp} - L^{-1}(1-\delta) \sum_{r=1}^{s} u_r y_{rj}^{\alpha} + \overline{\sigma}_p^{C} \phi_{1-\gamma}^{-1} - \overline{\varphi} \ge 0,\\
&\sum_{i=1}^{m} v_i x_{ip} - L^{-1}(1-\delta) \sum_{i=1}^{m} v_i x_{ip}^{\alpha} + \overline{\sigma}_p^{I} \phi_{1-\gamma}^{-1} \ge 1,\\
&\sum_{i=1}^{m} v_i x_{ip} + R^{-1}(\delta) \sum_{i=1}^{m} v_i x_{ip}^{\beta} - \overline{\sigma}_p^{I} \phi_{1-\gamma}^{-1} \le 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + R^{-1}(\delta_j) \sum_{r=1}^{s} u_r y_{rp}^{\beta} + L^{-1}(1-\delta_j) \sum_{i=1}^{m} v_i x_{ij}^{\alpha}) - \overline{\sigma}_j^{A} \phi_{1-\gamma_j}^{-1} \le 0, j = 1,\dots,n, \qquad (5)\\
&(\overline{\sigma}_p^{C})^2 = (\sum_{r=1}^{s} u_r^2\, \sigma_{rp}^2),\\
&(\overline{\sigma}_p^{I})^2 = (\sum_{i=1}^{m} v_i^2\, \sigma_{ip}^2),\\
&(\overline{\sigma}_p^{A})^2 = (\sum_{r=1}^{s} u_r^2\, \sigma_{rj}^2 + \sum_{i=1}^{m} v_i^2\, \sigma_{ij}^2), j = 1,\dots,n,\\
&u_r, v_i, \overline{\sigma}_p^{C}, \overline{\sigma}_p^{I}, \overline{\sigma}_p^{A} \ge 0, r = 1,\dots,s, i = 1,\dots,m, j = 1,\dots,n.
\end{aligned}
$$

**Theorem 2.** Let $\bar{\lambda}_1 = (m_1, \alpha_1, \beta_1)_{LR}$and $\bar{\lambda}_2 = (m_2, \alpha_2, \beta_2)_{LR}$ be two independent L-R type fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$:

- When $\alpha \le 0.5, \mathrm{Cr}\{\bar{\lambda}_1 \ge \bar{\lambda}_2\} \ge \alpha$ if and only if $m_1 + \beta_1 R^{-1}(2\alpha) \ge m_2 - \alpha_2 R^{-1}(2\alpha)$.
- When $\alpha > 0.5, \mathrm{Cr}\{\bar{\lambda}_1 \ge \bar{\lambda}_2\} \ge \alpha$ if and only if $m_1 - \alpha_1 L^{-1}(2(1-\alpha)) \ge m_2 + \beta_2 L^{-1}(2(1-\alpha))$.

For proof, see [36].

Finally, the probability-credibility CCR model for $\delta_j, \delta \le 0.5$ as follows:

$$
\begin{aligned}
&\max \overline{\varphi} \\
&\text{s.t.}\ \overline{\varphi} - \sum_{r=1}^{s} u_r y_{rp} - R^{-1}(2\delta) \sum_{r=1}^{s} u_r y_{rp}^{\beta} - \overline{\theta}_p^{O} \phi_{1-\gamma}^{-1} \le 0, \\
&\sum_{i=1}^{m} v_i (x_{ip} + R^{-1}(2\delta) x_{ip}^{\beta}) + \overline{\theta}_p^{I} \phi_{1-\gamma}^{-1} \ge 1, \\
&\sum_{i=1}^{m} v_i (x_{ip} - R^{-1}(2\delta) x_{ip}^{\alpha}) - \overline{\theta}_p^{I} \phi_{1-\gamma}^{-1} \le 1, \\
&\sum_{r=1}^{s} u_r (y_{rj} - R^{-1}(2\delta_j) y_{rj}^{\alpha}) - \sum_{i=1}^{m} v_i\,(x_{ij} + R^{-1}(2\delta_j) x_{ij}^{\beta}) - \bar{\lambda}_j \phi_{1-\gamma_j}^{-1} \le 0, j = 1, \ldots, n, \qquad \textbf{\textit{(6)}} \\
&(\theta_p^{O})^2 = \sum_{r=1}^{s} u_r^2 \operatorname{var}(y_{rp}^{m}), \\
&(\theta_p^{I})^2 = \sum_{i=1}^{m} v_i^2 \operatorname{var}(x_{ip}^{m}), \\
&(\bar{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2 \operatorname{var}(y_{rp}^{m}) + \sum_{i=1}^{m} v_i^2 \operatorname{var}(x_{ip}^{m}), j = 1, \ldots, n, \\
&u_r, v_i, \theta_p^{O}, \theta_p^{I}, \bar{\lambda}_j \ge 0, r = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}
$$

And for $\delta_j, \delta > 0.5$:

$$
\begin{aligned}
&\max \overline{\varphi} \\
&\text{s.t.}\ \overline{\varphi} - \sum_{r=1}^{s} u_r y_{rp} + L^{-1}(2(1-\delta)) \sum_{r=1}^{s} u_r y_{rj}^{\alpha} - \overline{\theta}_p^{O} \phi_{1-\gamma}^{-1} \leq 0, \\
&\sum_{i=1}^{m} v_i (x_{ip} - L^{-1}(2(1-\delta)) x_{ip}^{\alpha}) + \overline{\theta}_p^{I} \phi_{1-\gamma}^{-1} \geq 1, \\
&\sum_{i=1}^{m} v_i (x_{ip} + L^{-1}(2(1-\delta)) x_{ip}^{\beta}) - \overline{\theta}_p^{I} \phi_{1-\gamma}^{-1} \leq 1, \\
&\sum_{r=1}^{s} u_r (y_{rj} + L^{-1}(2(1-\delta_j)) y_{rp}^{\beta}) - \sum_{i=1}^{m} v_i (x_{ij} - L^{-1}(2(1-\delta_j)) x_{ij}^{\alpha}) - \overline{\lambda}_j \phi_{1-\gamma}^{-1} \leq 0, j = 1, \ldots, n, \\
&(\theta_p^{O})^2 = \sum_{r=1}^{s} u_r^2 \operatorname{var}(y_{rp}^{m}), \\
&(\theta_p^{I})^2 = \sum_{i=1}^{m} v_i^2 \operatorname{var}(x_{ip}^{m}), \\
&(\overline{\lambda}_j)^2 = (\sum_{r=1}^{s} u_r^2 \operatorname{var}(y_{rp}^{m}) + \sum_{i=1}^{m} v_i^2 \operatorname{var}(x_{ip}^{m})), j = 1, \ldots, n, \\
&u_r, v_i, \overline{\theta}_p^{O}, \overline{\theta}_p^{I}, \overline{\lambda}_j \geq 0, r = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}
\tag{7}
$$

**Definition 14.** A DMU  is said to be (probability-possibility, probability-necessity and probability-credibility) $\gamma$-efficient if the objective function of related models, $\varphi$, is greater than or equal to unity at the threshold level $1-\gamma$; otherwise, it is said to be (probability-possibility, probability-necessity and probability-credibility) $\gamma$-efficient.

In another study, Tavana et al. [44] developed an imprecise DEA-based formulation for dealing with the randomness of fuzzy variables on a possibility space $(\theta, P(\theta), \text{Pos})$ through efficiency measurement. Similar to the details as in the previous section, proposed model is as follows:

for $\delta > 0.5$:

$$
\begin{aligned}
&\max \varphi \\
&\text{s.t.}\ \varphi + \theta_o^O \phi^{-1}(\delta) \le \sum_{r=1}^{s} u_r (y_{ro}^{m_2} + R^{-1}(\gamma) y_{ro}^{\beta}), \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_2} + R^{-1}(\gamma) x_{io}^{\beta}) - \theta_o^I \phi^{-1}(\delta) \ge 1, \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_1} - L^{-1}(\gamma) x_{io}^{\alpha}) + \theta_o^I \phi^{-1}(\delta) \le 1, \\
&\sum_{r=1}^{s} u_r ( y_{rj}^{m_1} - L^{-1}(\gamma) y_{rj}^{\alpha}) - \sum_{i=1}^{m} v_i (x_{ij}^{m_2} + R^{-1}(\gamma) x_{ij}^{\beta}) + \phi^{-1}(\delta) \lambda_j \le 0, j = 1, \ldots, n, \\
&(\theta_o^O)^2 = \sum_{r=1}^{s} u_r^2\, (\hat{y}_{ro}^{m_1} - L^{-1}(\gamma) \hat{y}_{ro}^{\alpha}), \\
&(\theta_o^I)^2 = \sum_{i=1}^{m} v_i^2\, (\hat{x}_{io}^{m_1} - L^{-1}(\gamma) \hat{x}_{io}^{\alpha}), \\
&(\lambda_j)^2 = \sum_{r=1}^{s} u_r^2\, \hat{y}_{rj}^{m_2} + \sum_{i=1}^{m} v_i^2\, \hat{x}_{ij}^{m_2} - L^{-1}(\gamma) (\sum_{r=1}^{s} u_r^2\, \hat{y}_{rj}^{\beta} + \sum_{i=1}^{m} v_i^2\, \hat{x}_{ij}^{\beta}), j = 1, \ldots, n, \\
&u_r, v_i, \theta_p^O, \theta_p^I, \bar{\theta}_p^I, \lambda_j \ge 0, r = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}
\tag{8}
$$

And for $\delta \le 0.5$:

$$
\begin{aligned}
&\max \varphi \\
&\text{s.t.}\ \varphi - \sum_{r=1}^{s} u_r (y_{ro}^{m_2} + R^{-1}(\gamma) \bar{\tilde{y}}_{ro}^{\beta}) + \bar{\theta}_o^O \phi^{-1}(\delta) \le 0, \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_2} + R^{-1}(\gamma) x_{io}^{\beta}) - \bar{\theta}_o^I \phi^{-1}(\delta) \ge 1, \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_1} - L^{-1}(\gamma) x_{io}^{\alpha}) + \bar{\theta}_o^I \phi^{-1}(\delta) \le 1, \\
&\sum_{r=1}^{s} u_r ( y_{rj}^{m_1} - L^{-1}(\gamma) \sum_{r=1}^{s} u_r\, y_{rj}^{\alpha}) - \sum_{i=1}^{m} v_i (\bar{\tilde{x}}_{ip}^{m_2} + R^{-1}(\gamma) \sum_{i=1}^{m} v_i\, \bar{\tilde{x}}_{ip}^{\beta}) + \phi^{-1}(\delta) \bar{\lambda}_j \le 0, j = 1, \ldots, n, \\
&(\bar{\theta}_o^O)^2 = \sum_{r=1}^{s} u_r^2\, (\hat{y}_{ro}^{m_2} + R^{-1}(\gamma) \hat{y}_{ro}^{\beta}), \\
&(\bar{\theta}_o^I)^2 = \sum_{i=1}^{m} v_i^2\, (\hat{x}_{io}^{m_2} + R^{-1}(\gamma) \hat{x}_{io}^{\beta}), \\
&(\bar{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2\, \hat{y}_{rj}^{m_2} + \sum_{i=1}^{m} v_i^2\, \hat{x}_{ij}^{m_2} + R^{-1}(\gamma) (\sum_{r=1}^{s} u_r^2\, \hat{y}_{rj}^{\beta} + \sum_{i=1}^{m} v_i^2\, \hat{x}_{ij}^{\beta}), j = 1, \ldots, n, \\
&u_r, v_i, \bar{\theta}_p^O, \bar{\theta}_p^I, \lambda_j \ge 0, r = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}
\tag{9}
$$

Furthermore, they presented a Necessity-Probability constrained programming model under fuzzy probability necessity constraints as follow:

for $\delta > 0.5$:

$$\begin{aligned}
&\max \overline{\varphi} \\
&\text{s.t.}\ \overline{\varphi} - \sum_{r=1}^{s} u_r y_{ro}^{m_1} + L^{-1}(1-\gamma) \sum_{r=1}^{s} u_r y_{ro}^{\alpha} + \tilde{\theta}_o^{O} \phi^{-1}(\delta) \leq 0, \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_1} - L^{-1}(1-\gamma) x_{io}^{\alpha}) - \tilde{\theta}_o^{I} \phi^{-1}(\delta) \geq 1, \\
&\sum_{i=1}^{m} v_i x_{io}^{m_2} + R^{-1}(\gamma) \sum_{i=1}^{m} v_i\, x_{ip}^{\beta} + \tilde{\theta}_o^{I} \phi^{-1}(\delta) \leq 1, \\
&\sum_{r=1}^{s} u_r (y_{rj}^{m_2} + R^{-1}(\gamma) y_{rj}^{\beta}) - \sum_{i=1}^{m} v_i\, (x_{ij}^{m_1} - L^{-1}(1-\gamma) x_{ij}^{\alpha}) + \tilde{\lambda}_j \phi^{-1}(\delta) \leq 0, j = 1, \ldots, n, \qquad \textbf{\textit{(10)}} \\
&(\tilde{\theta}_o^{O})^2 = \sum_{r=1}^{s} u_r^2\, (\hat{y}_{ro}^{m_1} + R^{-1}(\gamma) \hat{y}_{ro}^{\alpha}), \\
&(\tilde{\theta}_o^{I})^2 = \sum_{i=1}^{m} v_i^2\, (\hat{x}_{io}^{m_2} + R^{-1}(\gamma) \hat{x}_{io}^{\beta}), \\
&(\tilde{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2\, \hat{y}_{rj}^{m_1} + \sum_{i=1}^{m} v_i^2\, \hat{x}_{ij}^{m_1}, j = 1, \ldots, n, \\
&u_r, v_i, \tilde{\theta}_o^{O}, \tilde{\theta}_o^{I}, \tilde{\lambda}_j \geq 0, r = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}$$

For $\delta \leq 0/5$:

$$
\begin{aligned}
&\max \overline{\varphi} \\
&\text{s.t.}\ \overline{\varphi} - \sum_{r=1}^{s} u_r y_{ro}^{m_1} + L^{-1}(1-\gamma) \sum_{r=1}^{s} u_r y_{ro}^{\alpha} + \hat{\theta}_o^{O} \phi^{-1}(\delta) \le 0, \\
&\sum_{i=1}^{m} v_i x_{io}^{m_1} - L^{-1}(1-\gamma) \sum_{i=1}^{m} v_i\, x_{io}^{\alpha} - \hat{\theta}_o^{I} \phi^{-1}(\delta) \ge 1, \\
&\sum_{i=1}^{m} v_i x_{io}^{m_2} + R^{-1}(\gamma) \sum_{i=1}^{m} v_i\, x_{ip}^{\beta} + \hat{\theta}_o^{I} \phi^{-1}(\delta) \le 1, \\
&\sum_{r=1}^{s} u_r (y_{rj}^{m_2} + R^{-1}(\gamma) y_{rj}^{\beta}) - \sum_{i=1}^{m} v_i\, (x_{ij}^{m_1} - L^{-1}(1-\gamma) x_{ij}^{\alpha}) + \hat{\lambda}_j \phi^{-1}(\delta) \le 0, j = 1, \ldots, n, \qquad \textbf{\textit{(11)}} \\
&(\hat{\theta}_o^{O})^2 = \sum_{r=1}^{s} u_r^2\, (\hat{y}_{ro}^{m_1} - L^{-1}(1-\gamma) \hat{y}_{ro}^{\alpha}), \\
&(\hat{\theta}_o^{I})^2 = \sum_{i=1}^{m} v_i^2\, (\hat{x}_{io}^{m_1} - L^{-1}(\gamma) \hat{x}_{io}^{\beta}), \\
&(\hat{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2\, \hat{y}_{rj}^{m_1} + \sum_{i=1}^{m} v_i^2\, \hat{x}_{ij}^{m_1} - L^{-1}(1-\gamma)(\sum_{i=1}^{m} v_i^2\, \hat{x}_{ij}^{\alpha} + \sum_{r=1}^{s} u_r^2\, \hat{y}_{rj}^{\alpha}), j = 1, \ldots, n, \\
&u_r, v_i, \hat{\theta}_o^{O}, \hat{\theta}_o^{I}, \hat{\lambda}_j \ge 0, r = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}
$$

In the models presented, Tavana et al. [44] discussed the CCR model in which the inputs and outputs are random-fuzzy parameters on a possibility space with a normal distribution. In general programs, the possibility levels may be unknown or imprecise, in particular for fuzzy and stochastic production sets. Hence, they developed the proposed models with the fuzzy threshold level to take into account the generalized DEA model. They assumed the fuzzy $\delta$, denoted by $\tilde{\delta} = (\delta^{\alpha}, \delta^{m_1}, \delta^{m_2}, \delta^{\beta})$ and the final model is as follows:

$$
\begin{aligned}
&\sum_{i=1}^{m} v_i\,(x_{ip}^{m_2} + x_{ip}^{\beta} R^{-1}(\gamma)) - 1 + \sum_{i=1}^{m} v_i^2\,(\hat{x}_{ip}^{m_2} + R^{-1}(\gamma)\hat{x}_{ip}^{\beta})\phi^{-1}(\tilde{\delta}^{\gamma R}) \le 0\\
&\max \overline{\varphi}\\
&\text{s.t.}\, \overline{\varphi} - \sum_{r=1}^{s} u_r y_{ro}^{m_2} + R^{-1}(1-\gamma)\sum_{r=1}^{s} u_r y_{ro}^{\beta} + \rho_o^{O}\phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^{\beta}) \le 0,\\
&\sum_{i=1}^{m} v_i x_{io}^{m_2} + R^{-1}(1-\gamma)\sum_{i=1}^{m} v_i\, x_{io}^{\beta} - \rho_o^{I}\phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^{\beta}) \le 1,\\
&\sum_{i=1}^{m} v_i x_{io}^{m_1} - R^{-1}(\gamma)\sum_{i=1}^{m} v_i\, x_{io}^{\alpha} - \rho_o^{I}\phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^{\beta}) \ge 1,\\
&\sum_{r=1}^{s} u_r y_{rj}^{m_2} - \sum_{i=1}^{m} v_i\, x_{ij}^{m_1} + R^{-1}(1-\gamma)(\sum_{r=1}^{s} u_r y_{rj}^{\beta} + \sum_{i=1}^{m} v_i\, x_{ij}^{\alpha}) + h_j\phi^{-1}(\delta^{m_2} + R^{-1}(\gamma)\delta^{\beta}) \le 0, j = \qquad (12)\\
&(\rho_o^{O})^2 = \sum_{r=1}^{s} u_r^2\,(\hat{y}_{ro}^{m_2} + R^{-1}(1-\gamma)\hat{y}_{ro}^{\beta}),\\
&(\rho_o^{I})^2 = \sum_{i=1}^{m} v_i^2\,(\hat{x}_{io}^{m_2} + R^{-1}(1-\gamma)\hat{x}_{io}^{\beta}),\\
&(h_j)^2 = \sum_{r=1}^{s} u_r^2\,(\hat{y}_{rj}^{m_2} + R^{-1}(1-\gamma)\hat{y}_{rj}^{\beta}) + \sum_{i=1}^{m} v_i^2\,(\hat{x}_{ij}^{m_2} + R^{-1}(1-\gamma)\hat{x}_{io}^{\beta}), j = 1,\ldots,n,\\
&u_r, v_i, \rho_o^{O}, \rho_o^{I}, h_j \ge 0, r = 1,\ldots,s, i = 1,\ldots,m, j = 1,\ldots,n.
\end{aligned}
$$

The $\mathrm{DMU_o}$is called necessity- probabilistic $\gamma - \tilde{\delta}$ efficient if $\varphi_o = 1$ where $\varphi_o = \overline{\varphi}_o/\max_j(\overline{\varphi}_j)$.

## 5. Numerical Example

In this section, we show the results of model (8) and model (9) with a numerical example. In this example, we consider 10 DMUs with two random-fuzzy inputs and one random-fuzzy output on a possibility space. The random-fuzzy inputs, $\bar{\tilde{x}}_{ij}$, and random-fuzzy outputs, $\bar{\tilde{y}}_{rj}$, are normally-distributed with triangular fuzzy means and triangular fuzzy variances as $\bar{\tilde{x}}_{ij} \sim N(\overline{x}_{ij}, \overline{\sigma}_{ij}^2)$ where $\overline{x}_{ij} = (x_{ij}^{m}, x_{ij}^{\alpha}, x_{ij}^{\beta})_{LR}$ and $\overline{\sigma}_{ij}^2 = (\hat{x}_{ij}^{\alpha}, \hat{x}_{ij}^{m}, \hat{x}_{ij}^{\beta})_{LR}$; and $\bar{\tilde{y}}_{rj} \sim N(\overline{y}_{rj}, \overline{\sigma}_{rj}^2)$ where $\overline{y}_{rj} = (y_{rj}^{m}, y_{rj}^{\alpha}, y_{rj}^{\beta})_{LR}$ and $\overline{\sigma}_{rj}^2 = (\hat{y}_{rj}^{\alpha}, \hat{y}_{rj}^{m}, \hat{y}_{rj}^{\beta})_{LR}$. The data are shown in ***Table 1***.

Using “what if” analysis in performance evaluation for the possibility-probability developed models, we first assume $\delta = 0.5$ when $\gamma$ assumes the four different measures of 0.1, 0.3, 0.5 and 0.9. We then assume $\gamma = 0.5$ when $\delta$ assumes the four different measures as 0.1, 0.3, 0.5 and 0.9. ***Table 2*** and ***Table 3*** report the results of models (8) and (9) for the above mentioned levels.

## 6. Conclusion

A DEA model basically draws three critical elements: The model specification, the reference set itself, and the definition of the production possibility set. Starting from the latter, the production possibility set can either be defined as complete and known (like in conventional DEA) or as potentially extending beyond or excluding the reference set (like in stochastic DEA). The reference set, the very observations that form the engine of the non-parametric approach, can be either precise (as in conventional DEA), outcomes of stochastic processes (as in stochastic frontier analysis), or imprecise (as in the fuzzy DEA models). Classic DEA generally uses deterministic data to evaluate performance, but today in the real world uncertainty is clearly observed in data. In this research, we first reviewed the data envelopment analysis data in a random field and then in a random fuzzy field simultaneously.

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